第四章、数字特征与特征函数

§4.1 数学期望

- 含义: 平均值.
- 时间平均: 大量观测值的算术平均.

X 的(独立的)观测值$a_1, a_2, \cdots, a_n$, 其平均值为

$$\frac{1}{n}(a_1 + \cdots + a_n).$$

- 空间平均: 所有可能值的加权平均(总和).

假设X 为离散型, 分布列为$P(X=x_k)=p_k, \forall k$. 那么,

$$a_1+\cdots+a_n=\sum_k x_k n_k,$$

$$n_k=|\{m:1\leqslant m\leqslant n, a_m=x_k\}|.$$

- 根据概率的频率含义:

$$\frac{n_k}{n}\approx p_k,$$

$$\text{因此,}\ \frac{1}{n}(a_1+\cdots+a_n)\approx\sum_k x_k p_k.$$

离散型

- 若$\sum_k |x_k| p_k < \infty$, 则称

$$\sum_k x_k p_k$$

为X 的(数学)期望, 记为EX. (期望存在, 定义4.1.1.)

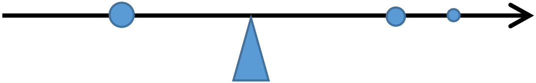
- 若$\sum_k |x_k| p_k = \infty$, 则说$X$ 的期望不存在. 例4.1.5.

$$x_k = (-1)^k \frac{2^k}{k}, \quad p_k = \frac{1}{2^k}, \quad k \geqslant 1.$$

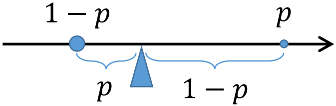
- 记

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0.$$

- 若$\sum_k x_k^{\pm} p_k$ 一个收敛而另一个发散, 也称$EX = \sum_k x_k p_k$ 为X 的期望. (期望不存在).
- 分布的数字特征: 若$X \overset{d}{=} Y$, 则$EX = EY$.
- 重心:



- 例4.1.1. Bernoulli 分布. $E1_A = P(A)$.
- 例4.1.3. 泊松分布.



(1) $X \geqslant 0$, EX 有意义.

(2) $x_k = k$:

$$kp_k = k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda p_{k-1}, \quad k \geqslant 1.$$

(3) 计算:

$$EX = \sum_{k=0}^{\infty} k \cdot p_k = \sum_{\ell=0}^{\infty} \lambda \frac{\lambda^\ell}{\ell!} e^{-\lambda} = \lambda.$$

- 习题四、7. X 取非负整数:

$$EX = \sum_{n=1}^{\infty} P(X \geqslant n) = \sum_{n=0}^{\infty} P(X > n).$$

- 证明:

$$EX = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} \sum_{n=1}^{k} p_k = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} p_k.$$

- 例4.1.4. 几何分布. $P(X > n) = q^n, \forall n \geqslant 0.$

$$EX = \sum_{n=0}^{\infty} q^n = \frac{q^0}{1-q} = \frac{1}{p}.$$

连续型.

- 离散逼近: $x_0 < x_1 < \cdots < x_n$,

$$\sum_i \underset{\sim\sim\sim}{x_i p(x_i)} \Delta x_i \to \int \underset{\sim\sim\sim}{xp(x)} dx.$$

- 若 $\int |x| p(x) dx < \infty$, 则称

$$\int xp(x)dx$$

为X 的(数学)期望, 记为EX. (期望存在, 定义4.1.2.)

- $\int x^{\pm} p(x)dx$ 不同时为∞ 时，

$$EX = \int xp(x)dx.$$

- 例4.1.13. 柯西分布，EX 无意义.

$$p(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}.$$

- 若$X \geqslant 0$, 则

$$EX = \int_0^{\infty} G(x)dx.$$

- 证明:

$$EX = \int_0^{\infty} xp(x)dx = -xG(x)|_0^{\infty} + \int_0^{\infty} G(x)dx,$$

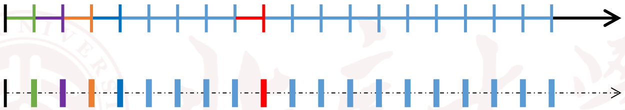
$$\text{其中,}\quad aG(a) \leqslant \int_a^{\infty} yp(y)dy.$$

- 例4.1.12. $X \sim \mathrm{Exp}(\lambda)$.

$$\int_0^{\infty} P(X > x)dx = \int_0^{\infty} e^{-\lambda x}dx = \frac{1}{\lambda}.$$

一般情形.

- 先假设$X \geqslant 0$. 微调位置: $Y = \frac{1}{n}\lceil nX \rceil$.



- 直观: $|X - Y| \leqslant \frac{1}{n}$, 因此$|EX - EY| \leqslant \frac{1}{n}$.
- 计算EY:

$$EY = \sum_{k=1}^{\infty} \frac{1}{n} P\left(Y \geqslant \frac{k}{n}\right) = \sum_{\ell=0}^{\infty} \frac{1}{n} P\left(X > \frac{\ell}{n}\right) \to \int_0^{\infty} G(x)dx.$$

- 或者微调位置: $\hat{Y} = Y - \frac{1}{n}$.

$$E\hat{Y} = \sum_{k=0}^{\infty} \frac{k}{n}\left(F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)\right) \to \int_0^{\infty} x dF(x).$$

- 一般的X: $\int x dF(x)$.

- Lebesgue-Stieltjes 积分: $\Delta = \max_i \Delta x_i$.

$$\int xdF(x) := \lim_{\Delta\to 0}\sum_i x_i\left(F(x_{i+1}) - F(x_i)\right).$$

- 若$\int |x|dF(x) < \infty$, 则称

$$\int xdF(x)$$

为X 的期望, 记为EX. (期望存在, 定义4.1.3.)

- 定义与离散型、连续型的一致.

- $X \geqslant 0$, 则

$$EX = \int_0^{\infty} P(X > x)dx.$$

若$EX^{\pm}$ 不全为∞, 则$EX := EX^+ - EX^-$. (期望有意义)

- 期望存在:

$$EX^{\pm} < \infty \quad \text{iff} \quad E|X| < \infty \quad \text{iff} \quad \int |x|dF(x).$$

- 若X 有界(i.e., $P(|X| \leqslant M) = 1$), 则期望存在.
- 期望是分布的数字特征.

- $X \equiv c$, 则$EX = c$.
- 单调性:

$$若X \geqslant Y, 则EX \geqslant EY.$$

- 线性:

$$E(aX) = aEX, \quad E(X+Y) = EX + EY.$$

- 若$X \geqslant 0$ 且$EX = 0$, 则$X = 0$.

(1) $P(X > \frac{1}{n}) = 0$:

$$0 = EX \geqslant EX1_{\{X > \frac{1}{n}\}} \geqslant E\frac{1}{n} \cdot 1_{\{X > \frac{1}{n}\}} = \frac{1}{n}P\left(X > \frac{1}{n}\right).$$

(2) $P(X > 0) = \lim_n P\left(X > \frac{1}{n}\right) = 0$.

- 若$X\geqslant 0$ 且$EX<\infty$, 则

$$\lim_{x\to\infty} xG(x)=\lim_{x\to\infty} EX\cdot 1_{\{X>x\}}=0.$$

(1) $xG(x)$:

$$\leqslant 2\int_{x/2}^{x} G(y)dy \leqslant 2\int_{x/2}^{\infty} G(y)dy \to 0.$$

(2) $EX\cdot 1_{\{X>x\}}$:

$$=\int_0^\infty P(**>y)dy=\int_0^\infty P(X>x,X>y)dy$$
$$=\int_0^x P(X>x)dy+\int_x^\infty P(X>y)dy.$$

- $X \geqslant 0$, 则

$$\lim_{x\to\infty} EX\cdot 1_{\{X\leqslant x\}} \to EX.$$

(1) $EX<\infty$: $EX\cdot 1_{\{X>x\}}\to 0$.

(2) $EX=\infty$: $\forall M$,

$$\begin{aligned} EX\cdot 1_{\{X\leqslant x\}} &\geqslant \int_0^M P(y<X\leqslant x)dy \\ &\geqslant \int_0^M P(X>y)dy - MP(X>x) \\ &\to \int_0^M P(X>y)dy. \quad (x\to\infty). \end{aligned}$$

令$M\to\infty$, 得

$$\lim_{x\to\infty} EX\cdot 1_{\{X\leqslant x\}} = \infty.$$

- 函数的期望: $Y = f(X)$, 则$EY = \int f(x)dF_X(x)$. (定理4.1.1, 了解)
- 离散型:

$$Ef(X) = \sum_k f(x_k)p_k. \quad (4.1.18)$$

- 连续型:

$$Ef(X) = \int f(x)p(x)dx. \quad (4.1.20)$$

- 高维:

$$Ef(\vec{X}) = \sum_k f(\vec{x}_k)p_k, \quad Ef(\vec{X}) = \int f(\vec{x})p(\vec{x})d\vec{x}.$$

- 相互独立则

$$E(XY) = (EX)(EY).$$

- 例. X, Y 相互独立且是连续型, 则

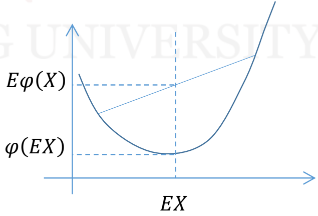
$$EXY = \iint xyp_X(x)p_Y(y)dxdy.$$

- Jensen不等式(4.3.7):

对任意凸函数φ,

$E\varphi(X) \geqslant \varphi(EX)$.

例. $EX^2 \geqslant (EX)^2$.



例4.1.11. $X \sim N(\mu, \sigma^2)$, 则$\mu = EX$.

- $Z \sim N(0,1)$, 期望存在, $Z \overset{d}{=} -Z$, 则

$$EZ = E(-Z) = -EZ \quad \Rightarrow \quad EZ = 0.$$

- $Z := \frac{X-\mu}{\sigma} \sim N(0,1)$.

$$EX = E(\mu + \sigma Z) = \mu.$$

- 一般地, 若$X - \mu \overset{d}{=} \mu - X$ 且$E|X| < \infty$, 则$EX = \mu$.

例4.1.16. 随机数目的期望.

- $X = 1_{A_1} + \cdots + 1_{A_n}$, 则$EX = \sum_i P(A_i)$.
- $X \sim B(n,p)$ 或$X \sim H(N,M,n)$, $EX = np$, $(p = \frac{M}{N})$.

$$X = X_1 + \cdots + X_n, \quad X_i \sim B(1,p).$$

- 匹配问题(例1.5.6) n 封信, n 个信封, 平均装对1 封.
- 例. 在有N 个顶点的完全图中, 每条边独立以概率p 涂红, $1-p$ 涂蓝, 得到X 个同色三角形.

$$EX = C_N^3 (p^3 + (1-p)^3).$$

条件期望

假设$E|Y| < \infty$.

- (X, Y) 是离散/连续型:

$$\varphi(x) := \sum_j y_j P(Y = y_j | X = x), \quad \int y p_{Y|X}(y|x) dy.$$

- $(\vec{X}, Y)$ 是离散/连续型, $E(Y|\vec{X}) = \varphi(\vec{X})$,

$$\varphi(\vec{x}) = \sum_j y_j P(Y = y_j | \vec{X} = \vec{x}), \quad \int y p_{Y|\vec{X}}(y|\vec{x}) dy.$$

- Y 关于X 的条件期望: $E(Y|X) := \varphi(X)$.
- 条件期望的线性:

$$E\big(a(X) + b(X)Y \big| X\big) = a(X) + b(X)E(Y|X).$$

重期望公式: $EY = EE(Y|X) = E\varphi(X)$. (4.2.59)

- 离散型:

$$E\varphi(X) = \sum_i \varphi(x_i)P(X = x_i)$$
$$= \sum_{i,j} y_j P(X = x_i, Y = y_j) = EY.$$

- 连续型:

$$E\varphi(X) = \int \varphi(x)p_X(x)dx = \iint y p_{X,Y}(x,y)dxdy = EY.$$

- $E\Big(E(Y|\vec{X},\vec{W})\Big|\vec{W}\Big) = E(Y|\vec{W})$.

$$\hat{E} = E(\cdot|\vec{W}), \quad E(Y|\vec{X},\vec{W}) = \hat{E}(Y|\vec{X}).$$

习题二、43. 每个虫卵独立地以概率p 孵化为幼虫.

虫卵数X 的期望存在, $Y=$ 幼虫数. 求EY.

- $\mathcal{L}(Y|X=n)=B(n,p)$:

$$E(Y|X=n)=np \Rightarrow E(Y|X)=Xp.$$

- $EY=EE(Y|X)=pEX.$
- $\xi=\xi_1,\xi_2,\cdots$ i.i.d., X 与它们独立, 期望都存在.

$$Y=\xi_1+\cdots+\xi_X \Rightarrow EY=(EX)\cdot(E\xi).$$

- 证: $\mathcal{L}\big(Y|X=n\big)=\mathcal{L}\big(\xi_1+\cdots+\xi_n\big).$

例(Polya 坛子). 最初有b 个黑球, r 个红球. 每次取一个, 放回并放入c 个同色球. B_n = “第n 个是黑球”, 则$P(B_n) = \frac{b}{b+r}$.

- 记X_n = “n 次后坛中黑球数”, $Y_n = \frac{X_n}{b+r+nc}$.
- 则

$$P(B_{n+1}|X_n = i) = \frac{i}{b+r+nc} =: y_i.$$

- 从而,

$$P(B_{n+1}) = \sum_i P(X_n = i) P(B_{n+1}|X_n = i)$$

$$= \sum_i P(Y_n = y_i) y_i = EY_n.$$

- $X_{n+1} = X_n + \underset{\sim\sim\sim\sim\sim}{c \cdot 1_{B_{n+1}}}$:

$$E(X_{n+1}|X_n) = X_n + c \cdot Y_n = Y_n(b + r + (n+1)c),$$
$$E(Y_{n+1}|X_n) = Y_n.$$

- 于是,

$$EY_{n+1} = EY_n = \cdots = EY_0 = \frac{b}{b+r}.$$

- $P(B_{n+1}) = EY_n = \frac{b}{b+r}, \forall n \geqslant 0.$

例. 假设$U_1, U_2, \cdots$ i.i.d. $\sim U(0,1)$. 记$S_n = U_1 + \cdots + U_n$. 求EX, 其中

$$X := \inf\{n : S_n \geqslant 1\}.$$

- 令$X_a := \inf\{n : S_n \geqslant a\}$. 记$f(a) = EX_a$. 待求$f(1)$.
- 分析X_a. 记$\hat{X}_b := \inf\{n-1 : U_2 + \cdots + U_n \geqslant b\}$, 则

$$X_a = 1_{\{U_1 \geqslant a\}} + 1_{\{U_1 < a\}}(1 + \hat{X}_{a-U_1}) = 1 + 1_{\{U_1 < a\}} \cdot \hat{X}_{a-U_1}.$$

- $\hat{X}_b$ 与U_1 相互独立, 且$\hat{X}_b \overset{d}{=} X_b$. 因此,

$$E[1_{\{U_1<a\}} \cdot \hat{X}_{a-U_1} | U_1 = x] = E(1_{\{x<a\}} \cdot \hat{X}_{a-x}) = 1_{\{x<a\}} \cdot f(a-x).$$

- $f(a) = 1 + \int_0^a f(y)dy$, $0 < a \leqslant 1$:

$$E(1_{\{U_1<a\}} \cdot f(a-U_1)) = \int_0^a f(a-x)dx.$$

- $f(a) = e^a$:

$$f'(a) = f(a) \Rightarrow (\ln f(a))' = 1 \Rightarrow f(a) = Ce^a.$$

因为$f(0) = 1$, 所以$C = 1$.

- $EX = f(1) = e$.

§4.2 方差、相关系数、矩

- 假设$E|X| < \infty$. 若

$$E(X - EX)^2$$

有限, 则称它为X 的方差(variance), 记为var(X) 或$D(X)$.

- 称

$$\sigma_X := \sqrt{\mathrm{var}(X)}$$

为X 的标准差/均方差(standard deviation). (定义4.2.1)

- 矩(moment): EX^k, $E(X - EX)^k$, Ee^{aX}. (定义4.2.5)

- 计算公式: 假设$E|X| < \infty$,

$$(X - EX)^2 = X^2 - 2(EX)\cdot X + (EX)^2,$$
$$\text{var}(X) = EX^2 - (EX)^2.$$

- 方差有限: $EX^2 < \infty$.
- 方差发散: $E|X| < \infty$, 但$EX^2 = \infty$
  $\text{var}(X) = E(X - EX)^2 = \infty$.
- 方差是分布的数字特征: 若$X \overset{d}{=} Y$ 则$\text{var}(X) = \text{var}(Y)$.
- 含义: 权重的分散程度.

$$\text{若 } \text{var}(X) = 0, \quad \text{则 } X = EX.$$

- 线性变换:

$$\mathrm{var}(a+bX) = E(bX-bEX)^2 = b^2\mathrm{var}(X).$$

- 标准化:

$$X^* = \frac{X-\mu}{\sigma}, \qquad EX^*=0,\ \mathrm{var}(X^*)=1.$$

例4.2.1. & 例4.2.2.

- $X \sim B(1,p)$, $X^2 = X$,

$$\mathrm{var}(X) = p - p^2 = pq.$$

- $Y \stackrel{d}{=} 1_{A_1} + \cdots + 1_{A_n}$.

(1) $Y^2 = \sum_{i,j} 1_{A_i} \cdot 1_{A_j} = \sum_{i,j} 1_{A_i A_j}$,

(2) $EY^2 = \sum_{i,j} P(A_i A_j)$.

$$\mathrm{var}(Y) = \sum_{i,j} P(A_i A_j) - \left(\sum_i P(A_i)\right)^2.$$

(3) 进一步, 假设$A_1, \cdots, A_n$ 两两独立. 记$p_i = P(A_i)$ 则

$$\mathrm{var}(Y) = \sum_i p_i + \sum_{i \neq j} p_i p_j - \sum_{i,j} p_i p_j = \sum_i p_i(1-p_i) = \sum_i \mathrm{var}(1_{A_i}).$$

例4.2.3. $X \sim P(\lambda)$.

- $EX = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda$.
- $\text{var}(X) = EX^2 - (EX)^2$.

(1) $EX(X-1)$:

$$\sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{\ell=0}^{\infty} \lambda^2 \frac{\lambda^\ell}{\ell!} e^{-\lambda} = \lambda^2.$$

(2) $\star\star = EX(X-1) + EX = \lambda^2 + \lambda$,

$$\text{var}(X) = \star\star - \lambda^2 = \lambda.$$

例4.2.4. $X \sim U(a,b)$.

- $U \sim U(0,1)$:

$$EU^2 = \int_0^1 x^2 dx = \frac{1}{3}, \quad \mathrm{var}(U) = \frac{1}{12}.$$

- $X \sim U(a,b)$,

$$U = \frac{X-a}{b-a} \sim U(0,1).$$

- $\mathrm{var}(X)$:

$$\mathrm{var}(a + (b-a)U) = \frac{(b-a)^2}{12}.$$

例4.2.5. $X \sim N(\mu, \sigma^2)$.

- 标准正态分布. $Z := X^* = \frac{X-\mu}{\sigma} \sim N(0,1)$.
- 二阶矩:

$$EZ^2 = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-\frac{x^2}{2}} dx = -\frac{2}{\sqrt{2\pi}} \int_0^\infty x de^{-\frac{x^2}{2}}$$
$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx = 1.$$

- $\mathrm{var}(X) = \mathrm{var}(\mu + \sigma Z) = \sigma^2 \mathrm{var}(Z) = \sigma^2$.

Chebyshev's 不等式 (4.2.10):

$$P(|X - EX| \geqslant \varepsilon) \leqslant \frac{\mathrm{var}(X)}{\varepsilon^2}, \quad \forall \varepsilon > 0.$$

证明技巧:

- $P(A) = E1_A$.
- 找$Y \geqslant 1_A$, 满足下式, 于是$P(A) \leqslant EY$.

$$Y \geqslant 0; \quad Y|_A \geqslant 1.$$

- 例. $Y = \frac{(X-EX)^2}{\varepsilon^2}$; $Y = \frac{|X-EX|^r}{\varepsilon^r}$. (Markov 不等式)(5.2.22)
- 例. 若$X \geqslant 0$, 则$P(X \geqslant C) \leqslant EX/C$.
- 例. $P(X \geqslant C) \leqslant Ee^{a(X-C)}$, 其中$a > 0$.

协方差、相关系数

- 假设$EX^2, EY^2 < \infty$. (定义4.2.3) 称

$$E(X - EX)(Y - EY).$$

为X, Y 的协方差(covariance), 记为$\text{cov}(X, Y)$ 或$\sigma_{X,Y}$.

- 和的方差:

$$\begin{aligned}\text{var}(X+Y) =& E\Big((X+Y) - (EX+EY)\Big)^2 \\ =& \text{var}(X) + \text{var}(Y) + 2E(X-EX)(Y-EY).\end{aligned}$$

- 相互独立则 $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$.

- 计算:

$$\begin{aligned}&(X-EX)(Y-EY)\\=&XY-(EX)\cdot Y-X\cdot(EY)+(EX)\cdot(EY).\end{aligned}$$

- 于是,

$$\mathrm{cov}(X,Y)=EXY-EXEY.$$

- 对称、双线性函数, $\tilde{X}=aX+c$, $\tilde{Y}=bY+d$:

$$\mathrm{cov}(\tilde{X},\tilde{Y})=ab\cdot\mathrm{cov}(X,Y).$$

Cauchy-Schwarz's 不等式 (定理4.2.1):

$$(EXY)^2 \leqslant EX^2EY^2.$$

- 不妨设$0 < EX^2, EY^2 < \infty$. 期望存在: $|XY| \leqslant \frac{1}{2}(X^2 + Y^2)$.
- 二次函数:

$$f(t) = E(tX + Y)^2 = (EX^2)t^2 + 2(EXY)t + EY^2 \geqslant 0.$$

- 等号成立 iff $f(t_0) = 0$ iff $Y = -t_0X$.

$$t_0 = -\frac{EXY}{EX^2}, \quad f(t_0) = \frac{EX^2EY^2 - (EXY)^2}{EX^2}.$$

- Hölder's 不等式: $0 < k, \ell < \infty$, $\frac{1}{k} + \frac{1}{\ell} = 1$,

$$EXY \leqslant (E|X|^k)^{1/k}(E|Y|^\ell)^{1/\ell}.$$

- 假设$0<\sigma_X^2,\sigma_Y^2<\infty$. 称

$$\rho_{X,Y}:=\frac{\sigma_{X,Y}}{\sigma_X\sigma_Y}=\frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}.$$

为X,Y 的(线性)相关系数, 简记为ρ. (定义4.2.3)

- 若$\tilde{X}=aX+c,\ \ \tilde{Y}=bY+d$, 则

$\rho_{\tilde{X},\tilde{Y}}=\rho_{X,Y}\quad(ab>0)$ 或$-\rho_{X,Y}\quad(ab<0)$.

$$\rho_{X,Y}=\rho_{X^*,Y^*}=\mathrm{cov}(X^*,Y^*)=EX^*Y^*.$$

- $|\rho|\leqslant 1$:

$$\rho=1 \text{ iff } Y^*=X^* \text{ iff } Y=aX+b,\ \ a>0;$$

$$\rho=-1 \text{ iff } Y^*=-X^* \text{ iff } Y=aX+b,\ \ a<0.$$

$$\rho=EX^*Y^*=\langle X^*,Y^*\rangle=\cos\theta.$$

- 不、正、负相关:

$$\operatorname{cov}(X, Y) = 0, \quad > 0, \quad < 0.$$

- 完全正、负相关 (定义4.2.4):

$$\rho = 1, \quad -1.$$

- 独立则线性不相关.
- 反之不然! 例:

$$X \sim N(0, 1), \; Y = X^2.$$

例4.2.8. $U \sim U(0,2\pi)$. $X = \cos U$, $Y = \cos(U+a)$.

- $Y = \cos \hat{U} \overset{d}{=} X, \quad \hat{U} \sim U(0,2\pi)$.
- 期望、方差: $EX = 0, \quad EX^2 = \frac{1}{2}$.

$$EX^2 = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2}$$

- 协方差、相关系数. $\rho_{X,Y} = \cos a$:

$$\mathrm{cov}(X,Y) = EXY = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\cos(\theta+a)d\theta = \frac{1}{2}\cos a.$$

- $a = 0$: $\quad Y = X, \quad \rho = 1$, 完全正相关.

  $a = \pi$: $\quad Y = -X, \quad \rho = -1$, 完全负相关.
- $a = \frac{\pi}{2}$ 或 $\frac{3\pi}{2}$: $\quad \rho = 0$, 不相关, 但是不独立.

例. 假设A, B 是(可测)事件.

- A, B 不、正、负相关: $P(AB)$ =、>、< $P(A)P(B)$.

$$\operatorname{cov}(1_A, 1_B) = E1_A1_B - E1_AE1_B = P(AB) - P(A)P(B).$$

- A, B 不相关 iff A 与B 相互独立. (性质4.2.5)
- 例. A = "产品一是次品", B = "产品二是次品".

  放回抽样, 不相关(独立) vs 不放回抽样, 负相关.

- $|P(AB) - P(A)P(B)| \leqslant \frac{1}{4}$. (4.2.31), 习题一45.

$$\rho_{A,B} := \frac{P(AB) - P(A)P(B)}{\sqrt{P(A)(1 - P(A))} \cdot \sqrt{P(B)(1 - P(B))}}. \quad (4.2.30)$$

例. 二元正态. 设(X,Y) 密度如下, 则$\rho_{X,Y}=\rho$ (4.2.28).

$$p(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left(u^2-2\rho uv+v^2\right)},$$

其中, $u=\frac{x-\mu_1}{\sigma_1}$, $v=\frac{y-\mu_2}{\sigma_2}$.

- 由定理3.2.1, $X\sim N(\mu_1,\sigma_1^2)$, $Y\sim N(\mu_2,\sigma_2^2)$.
  故$U:=X^*$, $V:=Y^*$ 的密度为$\hat{p}(u,v)$.
- $\rho_{X,Y}=EX^*Y^*=\rho$. (4.2.28)

$$=\iint u\cdot v\cdot\hat{p}(u,v)dudv$$

$$=\int\int u\cdot v\cdot\frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left((u-\rho v)^2+(1-\rho^2)v^2\right)}dudv$$

$$=\int\rho v\cdot v\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv=\rho EZ^2=\rho.$$

- 假设(X,Y) 服从二元正态分布. 则

X, Y 不相关$(\rho = 0)$ iff 相互独立$(p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y))$.

- 例4.2.9. $X, Y \sim N(0,1)$, $\rho_{X,Y} = 0$, 但不独立.

$$p(x,y) = p(x)p(y) + \frac{1}{2\pi} e^{-\pi^2} g(x)g(y)$$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad g(x) = \cos x \cdot 1_{\{|x|<\pi\}}.$$

- $\vec{X} = (X_1, \cdots, X_n)^T$ 的数字特征:

  期望: $E\vec{X} = (EX_1, \cdots, EX_n)^T$,

  协方差矩阵: $\mathbf{\Sigma} = (\sigma_{ij})_{n\times n}$, $\sigma_{ij} = \text{cov}(X_i, X_j)$. (4.2.18).

- $\mathbf{\Sigma}$ 半正定: 对称,

$$\vec{x}^T\mathbf{\Sigma}\vec{x} = \sum_{i,j}\sigma_{ij}x_ix_j = \sum_{i,j}x_ix_jE(X_i-\mu_i)(X_j-\mu_j)$$

$$= E\left(\sum_i x_i(X_i-\mu_i)\right)^2.$$

- $\mathbf{\Sigma}$ 正定指:

$$\sum_i x_i(X_i-\mu_i) = 0, \ \text{(a.s.)} \quad \Rightarrow \quad x_1 = \cdots = x_n = 0.$$

- 因此, $\mathbf{\Sigma}$ 正定iff $1, X_1, \cdots, X_n$ 线性无关.

最优预测1. 找$\hat{y} \in \mathbb{R}$ 使得$Q(\cdot)$ 达到最小，

$$Q(v) := E(Y - v)^2, \quad \forall v \in \mathbb{R}.$$

- 分解：

$$Y - v = (Y - \hat{y}) + (\hat{y} - v), \quad \forall v.$$

- 目标：交叉项为0.

$$Q(v) = Q(\hat{y}) + (\hat{y} - v)^2 + 2\underbrace{(\hat{y} - v) \cdot E(Y - \hat{y})}.$$

- 取$\hat{y} = EY$ 即可．(4.2.8).
- 上述分解为正交分解，且取$v = 0$ 知

$$E(Y - \hat{y})^2 = EY^2 - E\hat{y}^2 = \mathrm{var}(Y).$$

最优预测2. 找$\hat{a}, \hat{b} \in \mathbb{R}$ 使得

$$Q(a,b) := E\big(Y - (a+bX)\big)^2, \quad a, b \in \mathbb{R}$$

达到最小. 其中, $EX = 0$, $EX^2 = 1$.

- $\hat{a} = E(Y - \hat{b}X) = EY$: $Y - (a+bX) = (Y - bX) - a$.
- 令$Y_0 = Y - EY$, 问题化为找$\hat{b}$ 使得$f(\cdot)$ 达到最小.

$$f(b) = Q(\hat{a}, b) = E(Y_0 - bX)^2.$$

- (正交)分解及其交叉项.

$$Y_0 - bX = (Y_0 - \hat{b}X) \bigoplus (\hat{b} - b)X.$$

$$f(b) = f(\hat{b}) + (\hat{b} - b)^2 + 2(\hat{b} - b) \cdot E(Y_0 - \hat{b}X)X.$$

- 取$\hat{b} = EXY_0 = \text{cov}(X, Y)$ 即可.
- $E\big(Y - \hat{a} - \hat{b}X\big)^2 = E{Y_0}^2 - E(\hat{b}X)^2 = \Big(1 - \rho_{X,Y}^2\Big)\, \sigma_Y^2$.

  特别地, $\rho_{X,Y} = \pm 1$ 当且仅当$Y = \hat{a} + \hat{b}X$.

最优预测3*. 找$\hat{\psi}(\cdot)$ 使得$Q(\cdot)$ 达到最小.

$$Q(\psi)=E(Y-\psi(X))^2,\quad \psi(\cdot)\text{ 使得 } E\psi(X)^2<\infty.$$

- 正交分解:

$$Y-\psi(X)=(Y-\hat{\psi}(X))\bigoplus(\hat{\psi}(X)-\psi(X)).$$

- 根据条件期望的线性与重期望公式,

$$E\star\star=E\big(E(\star\star|X)\big)=E\Big(\star\big(E(Y|X)-\hat{\psi}(X)\big)\Big).$$

- 取$\hat{\psi}(X)=E(Y|X)=\varphi(X)$ 即可. 进一步,

$$E\big(Y-\varphi(X)\big)^2=EY^2-E\varphi(X)^2.$$

§4.4 母函数

注：在本节中, 随机变量都取非负整数值.

- 设$P(X=k)=p_k,\ k\geqslant 0$. 称

$$\sum_{k=0}^{\infty} p_k s^k, \quad s\in[-1,1]$$

为X 的母函数(generating function), 记为$g_X(s)$ 或$g(s)$.

- 表达式:

$$g(s)=Es^X=p_0+p_1 s+p_2 s^2+\cdots+p_k s^k+\cdots.$$

- 母函数刻画分布: $X\overset{d}{=}Y$ iff $g_X=g_Y$.

$$g^{(k)}(0)=p_k\cdot k!, \quad \forall k\geqslant 0.$$

- $g(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_k s^k + \cdots = Es^X$.
- 矩: 对$s \in (-1,1)$,

$$\begin{aligned} g'(s) =& p_1 + 2p_2 s + \cdots + kp_k s^{k-1} + \cdots = EXs^{X-1}, \\ g''(s) =& 2p_2 + \cdots + k(k-1)p_k s^{k-2} + \cdots = EX(X-1)s^{X-2}. \\ g^{(\ell)}(s) =& EX(X-1)\cdots(X-\ell+1)s^{X-\ell}. \end{aligned}$$

- 例,

$$\begin{aligned} EX =& g'(1) := \lim_{s\to 1-} g'(s), (EX < \infty \text{ or } EX = \infty\text{都成立}) \\ EX^2 =& g''(1) + g'(1). \end{aligned}$$

- 例4.4.4. $X \sim G(p)$.

$$g(s) = \sum_{k=1}^{\infty} q^{k-1} p s^k = \frac{ps}{1-qs}.$$

- 乘积: 若X 与Y 独立, 则$g_{X+Y}(s) = g_X(s)g_Y(s)$.

$$Es^{X+Y} = Es^X s^Y = Es^X \cdot Es^Y.$$

- 例4.4.5. $X \sim B(n,p)$.

$$g_n(s) = (q+ps)^n.$$

- 例4.4.6. $X \sim P(\lambda)$.

$$g_\lambda(s) = \sum_{k=0}^{\infty} s^k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda(s-1)}.$$

- 复合: 设$\xi=\xi_1,\xi_2,\cdots$ i.i.d., 且它们与W 独立.

$$令X=\xi_1+\cdots+\xi_W,\ 则g_X=g_W\circ g_\xi.$$

(1) 条件期望:

$$E(s^X|W=k)=E(s^{\xi_1+\cdots+\xi_k}|W=k)=E\star=g_\xi(s)^k.$$

(2) 重期望公式:

$$Es^X=E(E(s^X|W))=E\left(g_\xi(s)^W\right)=g_W\left(g_\xi(s)\right)=g_W\circ g_\xi(s).$$

- 期望:

$$EX=g'_X(1)=g'_W(g_\xi(1))g'_\xi(1)=EW\cdot E\xi.$$

$$EX=E(E(X|W))=E(W\cdot E\xi)=EW\cdot E\xi.$$

- 复合泊松分布: 若$W \sim P(\lambda)$, 则 $X = \xi_1 + \cdots + \xi_W$ 的母函数为

$$g_X(s) = \exp\{\lambda(g_\xi(s) - 1)\}.$$

称X 服从复合泊松分布.

- 例4.4.10. $W \sim P(\lambda)$, $\xi \sim B(1,p)$.

$$g_X(s) = \exp\{\lambda(q + ps - 1)\} = e^{\lambda p(s-1)}.$$

- 凸组合: X, Y, ξ 相互独立, $\xi \sim B(1,p)$. 令

$$W = X \cdot 1_{\{\xi=1\}} + Y \cdot 1_{\{\xi=0\}},$$

则 $g_W = p \cdot g_X + (1-p) \cdot g_Y$.

§4.5 特征函数

- 称

$$Ee^{itX} = E\cos(tX) + \sqrt{-1}E\sin(tX), \quad \forall t \in \mathbb{R}$$

为X 的特征函数(characteristic function), 记为$f_X(t)$ 或$f(t)$.

- $x + iy = (x, y)$. $\quad e^{itX} = (\cos(tX), \sin(tX))$.
- 基本性质1. $f(0) = 1$.
- $\|f(t)\| = \|Ee^{itX}\| \leqslant E\|e^{itX}\| = 1$.

$$\varphi : x + iy \mapsto \|x + iy\| = \sqrt{x^2 + y^2} \quad \text{是凸函数.}$$

- 基本性质2. f 一致连续:

(1) $\|\cdot\|$ 凸:

$$\|f(t+\varepsilon)-f(t)\| \leqslant E\|e^{i(t+\varepsilon)X}-e^{itX}\| = E\|e^{i\varepsilon X}-1\|.$$

(2) $\forall M>0$,

$$EY = EY\cdot 1_{\{|X|\leqslant M\}} + EY\cdot 1_{\{|X|>M\}}.$$

(3) 取M 使得$P(|X|>M)<\frac{\delta}{4}$, 则

$$\star\star \leqslant 2\cdot P(|X|>M) < \frac{\delta}{2},$$

$$\star\star \leqslant \max_{|x|\leqslant M}\|e^{i\varepsilon x}-1\| \leqslant \varepsilon M < \frac{\delta}{2}, \quad (\text{取}\varepsilon<\frac{\delta}{2M}).$$

- 基本性质3. f 半正定:

$\forall t_1,\cdots,t_n\in\mathbb{R}$, 令$a_{kj}=f(t_k-t_j)$, 则

$$\mathbf{A}=(a_{kj})_{n\times n}\text{ 半正定,}$$

$$\text{i.e.}\quad \text{(i) } \bar{\mathbf{A}}^T=\mathbf{A},\qquad \text{(ii) } \sum_{k,j}a_{kj}\lambda_k\overline{\lambda_j}\geqslant 0,\ \forall\lambda_1,\cdots,\lambda_n\in\mathbb{C}.$$

- 验证(i): 记$t=t_j-t_k$.

$$f(-t)=Ee^{i(-t)X}=E\overline{e^{itX}}=\overline{f(t)}.\quad(4.5.13)$$

- 验证(ii):

$$\sum_{k,j}a_{kj}\lambda_k\overline{\lambda_j}=\sum_{k,j}Ee^{i(t_k-t_j)X}\lambda_k\overline{\lambda_j}=E\left\|\sum_j\lambda_je^{it_jX}\right\|^2\geqslant 0.$$

- Bochner-Khinchine定理(定理5.2.9): 若$f:\mathbb{R}\to\mathbb{C}$ 满足 $f(0)=1$, 连续, 半正定, 则存在X 使得$f=f_X$.
- 逆转公式& 唯一性(定理4.5.1~ 4.5.2):

$$F(x)-F(y)=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ity}-e^{-itx}}{it}f(t)dt,\quad \forall x,y\in C(F).$$

- (定理4.5.3) 假设$\int\|f(t)\|dt<\infty$. 则相应分布函数F连续可导，且

$$p(x)=F'(x)=\frac{1}{2\pi}\int e^{-itx}f(t)dt.\quad (4.5.25).$$

- 性质4. 乘积: X 与Y 独立, 则$f_{X+Y}(t) = f_X(t)f_Y(t)$.

  (1) $X \sim B(n,p)$, $f_n(t) = (q + pe^{it})^n = (1 + p(e^{it} - 1))^n$.

  (2) $X \sim P(\lambda)$, $f_\lambda(t) = e^{\lambda(e^{it}-1)}$.

- 凸组合: 设X, Y, ξ 相互独立, $\xi \sim B(1,p)$.
  令$W = X\xi + Y(1-\xi)$, 则$f_W = pf_X + (1-p)f_Y$.

- 性质5. 若EX^k 存在, 则$f^{(k)}(0) = i^k EX^k$, 且

$$f(t) = 1 + f'(0)t + \frac{f''(0)}{2!}t^2 + \cdots + \frac{f^{(k)}(0)}{k!}t^k + o(t^k).$$

- 性质6. $f_{aX+b}(t) = Ee^{itaX+itb} = e^{itb}f_X(at)$.

例4.5.5. $X \sim N(\mu, \sigma^2)$.

- $Z = X^* \sim N(0,1)$:

$$f_Z(t) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2} + itx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-it)^2 + \frac{1}{2}(it)^2} dx = e^{-\frac{t^2}{2}}.$$

(严格计算见书)

- $X = \mu + \sigma Z$:

$$f_X(t) = Ee^{it(\mu+\sigma Z)} = e^{it\mu - \frac{1}{2}\sigma^2 t^2}.$$

随机向量的特征函数:

- 联合特征函数:

$$f_{\vec{X}}(\vec{t}) = Ee^{i\vec{t}\cdot\vec{X}} = Ee^{i(t_1X_1+\cdots+t_nX_n)}.$$

- 逆转公式: $f_{\vec{X}}(\vec{t}) \to F_{\vec{X}}(\vec{x})$.
- 若干性质(见书).
- 边缘: 例,

$$f_X(t) = f_{X,Y}(t, 0).$$

- X 与Y 独立 iff $f_{X,Y}(t, s) = f_X(t)f_Y(s)$.
- X 与Y 独立$\Rightarrow f_{X+Y}(t) = f_X(t)f_Y(t)$.

  反之不然(习题四、50).

§4.6 多元正态分布

- $\vec{\mu} \in \mathbb{R}^n$ (列向量), $\mathbf{\Sigma}$ 为$n \times n$ 的正定矩阵. 记

$$\vec{X} = (X_1, \cdots, X_n)^T \sim N(\vec{\mu}, \mathbf{\Sigma}).$$

- 联合密度:

$$p_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{2\pi}^n \sqrt{|\mathbf{\Sigma}|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \mathbf{\Sigma}^{-1} (\vec{x}-\vec{\mu})\right\}.$$

- n 维标准正态$\vec{Z} = (Z_1, \cdots, Z_n)^T \sim N(\vec{0}, \mathbf{I})$. 联合密度:

$$p_{\vec{Z}}(\vec{z}) = \frac{1}{\sqrt{2\pi}^n} \exp\left\{-\frac{1}{2}\|\vec{z}\|^2\right\}.$$

等价地, $Z_1, \cdots, Z_n$ i.i.d., $\sim N(0, 1)$.

- 设$\vec{Z} \sim N(\vec{0}, \mathbf{I})$, $\mathbf{A}$ 非退化, 则

$$\vec{Y} := \vec{\mu} + \mathbf{A}\vec{Z} \sim N(\vec{\mu}, \mathbf{\Sigma}), \quad (\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T.)$$

- 证: $p_{\vec{Y}}(\vec{y}) = p_{\vec{Z}}(\vec{z})|\frac{\partial \vec{z}}{\partial \vec{y}}| = C\exp\{-\frac{1}{2}\|\vec{z}\|^2\}$.

$$\|\vec{z}\|^2 = \|\mathbf{A}^{-1}(\vec{y}-\vec{\mu})\|^2 = (\vec{y}-\vec{\mu})^T\mathbf{A}^{-1,T}\mathbf{A}^{-1}(\vec{y}-\vec{\mu})$$
$$=(\vec{y}-\vec{\mu})^T(\mathbf{A}\mathbf{A}^T)^{-1}(\vec{y}-\vec{\mu}).$$

- 设$\vec{X} \sim N(\vec{\mu}, \mathbf{\Sigma})$. 取$\mathbf{A} = \sqrt{\mathbf{\Sigma}}$. 则

$$\vec{X} \stackrel{d}{=} \vec{Y} \Rightarrow \vec{V} := \mathbf{A}^{-1}(\vec{X}-\vec{\mu}) \stackrel{d}{=} \vec{Z}.$$

- 因此, $\exists\vec{V}$ 满足:

$$\vec{V} \sim N(\vec{0}, \mathbf{I}), \quad \vec{X} = \vec{\mu} + \mathbf{A}\vec{V}.$$

- 进一步, $\vec{X}$ 的非退化线性变换都服从n 维正态分布.

高斯分布与特征函数

- $N(\vec{0}, \mathbf{I})$:

$$f_{\vec{Z}}(\vec{t}) = Ee^{i\vec{t}\cdot\vec{Z}} = \prod_{k=1}^{n} Ee^{it_k Z_k} = \prod_{k=1}^{n} e^{-t_k^2/2} = e^{-\frac{1}{2}\|\vec{t}\|^2}.$$

- $N(\vec{\mu}, \mathbf{\Sigma})$: $\vec{X} = \vec{\mu} + \mathbf{A}\vec{Z}$, $\quad \mathbf{A} = \sqrt{\mathbf{\Sigma}}, \ \mathbf{A}\mathbf{A}^T = \mathbf{\Sigma}.$

$$\begin{aligned} f_{\vec{X}}(\vec{t}) &= Ee^{i\vec{t}\cdot(\vec{\mu}+\mathbf{A}\vec{Z})} = e^{i\vec{t}\cdot\vec{\mu}} Ee^{i(\mathbf{A}^T\vec{t})\cdot\vec{Z}} \\ &= e^{i\vec{t}\cdot\vec{\mu}} e^{-\frac{1}{2}\|\mathbf{A}^T\vec{t}\|^2} = \exp\left\{ i\vec{t}\cdot\vec{\mu} - \frac{1}{2}\vec{t}^T\mathbf{\Sigma}\vec{t} \right\}. \end{aligned}$$

- 设$\vec{\mu} \in \mathbb{R}^n$, $\mathbf{\Sigma}_{n\times n}$ 半正定. 若$\vec{X}$ 的联合特征函数为$\star$, 则称$\vec{X}$ 服从高斯分布$N(\vec{\mu}, \mathbf{\Sigma})$. 也称$\vec{X}$ 为一个高斯向量.
- $\vec{\mu} + \mathbf{A}\vec{Z} \sim N(\vec{\mu}, \mathbf{\Sigma})$, $\ \mathbf{A} = \sqrt{\mathbf{\Sigma}}$. $\qquad \mathbf{\Sigma}$ 非退化vs 退化.
- 高斯向量的任意线性变换仍服从高斯分布.

数字特征

- 假设$\vec{X} \sim N(\vec{\mu}, \mathbf{\Sigma})$. (其中$\mathbf{\Sigma}$ 半正定.)
- $\vec{X} \stackrel{d}{=} \vec{\mu} + \mathbf{A}\vec{Z} =: \vec{Y}$, 其中$\mathbf{A} = (a_{ij})_{n\times n} = \sqrt{\mathbf{\Sigma}}$.
- 期望: $E\vec{X} = \vec{\mu}$.

$$EX_i = EY_i = E(\mu_i + a_{i1}Z_1 + \cdots + a_{in}Z_n) = \mu_i.$$

- 协方差矩阵: $(\sigma_{ij}) = (\operatorname{cov}(X_i, X_j)) = \mathbf{\Sigma}$.

$$\operatorname{cov}(X_i, X_j) = E\sum_k a_{ik}\underline{Z_k}\sum_\ell a_{j\ell}\underline{Z_\ell} = \sum_{k,\ell} a_{ik}a_{j\ell} \times \underline{EZ_kZ_\ell}$$

$$= \sum_k a_{ik}a_{jk} = \sigma_{ij}.$$

定理 (定理4.6.6)

$\vec{X} \sim N(\vec{\mu}, \mathbf{\Sigma})$ iff $\forall a_1, \cdots, a_n \in \mathbb{R}$, $Y := \sum_{k=1}^n a_k X_k \sim N(\mu, \sigma^2)$.

- $\Rightarrow$:

$$Ee^{itY} = Ee^{i(ta_1,\cdots,ta_n)\cdot\vec{X}} = \exp\left\{it\sum_{k=1}^n a_k\mu_k - \frac{1}{2}t^2\left(\vec{a}^T\mathbf{\Sigma}\vec{a}\right)\right\}.$$

- $\Leftarrow$: (1) 数字特征: 根据假设有$EX_i^2 < \infty$. 令$E\vec{X} = \vec{\mu}$; $\vec{X}$ 的协方差阵为$\mathbf{\Sigma}$. 从而

$$\mu = EY = \sum_{k=1}^n a_k\mu_k, \quad \sigma^2 = \mathrm{cov}(Y, Y) = \vec{a}^T\mathbf{\Sigma}\vec{a}.$$

- (2) 特征函数:

$$Ee^{i\vec{t}\cdot\vec{X}} = Ee^{i\cdot 1\cdot\sum_{k=1}^n t_k X_k} = \exp\left\{i\mu - \frac{1}{2}\sigma^2\right\} \quad (\vec{a} = \vec{t})$$

$$= \exp\left\{i\vec{t}\cdot\vec{\mu} - \frac{1}{2}\vec{t}^T\mathbf{\Sigma}\vec{t}\right\}.$$

- 改写:

$$\vec{X} = (Y_1, \cdots, Y_r;\ W_{r+1}, \cdots, W_n)^T,$$
$$\vec{\mu} = (\nu_1, \cdots, \nu_r;\ w_{r+1}, \cdots, w_n)^T.$$
$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}.$$

- $\vec{Y} \sim N(\vec{\nu}, \mathbf{\Sigma}_{11})$:

$$f_{\vec{Y}}(\vec{s}) = f_{\vec{X}}(\vec{s}; \vec{0}) = \exp\left\{ i\vec{s} \cdot \vec{\nu} - \frac{1}{2}\vec{s}^T \mathbf{\Sigma}_{11} \vec{s} \right\}.$$

- Y_i 与W_k 不相关, $\forall i \leqslant r < k$ (即$\mathbf{\Sigma}_{12} = 0$) $\Leftrightarrow \vec{Y}$ 与$\vec{W}$ 独立:

$$\mathbf{\Sigma}_{12} = 0 \quad \Rightarrow \quad f_{\vec{X}}(\vec{s}; \vec{u}) = f_{\vec{Y}}(\vec{s}) f_{\vec{W}}(\vec{u}).$$

$\Sigma_{n\times n}$ 退化的情形. 假设$\vec{X} \sim N(\vec{0}, \Sigma)$.

- 结论: 存在$\vec{V} \sim N(\vec{0}, \mathbf{I}_{n\times n})$ 以及$\mathbf{M}_{n\times n}$ 使得$\vec{X} = \mathbf{M}\vec{V}$.
- 记$\mathbf{A} = (a_{ij})_{n\times n} = \sqrt{\Sigma}$, 取$\vec{Z} \sim N(\vec{0}, \mathbf{I}_{n\times n})$, 则

$$\vec{X} \overset{d}{=} \mathbf{A}\vec{Z} =: (Y_1, \cdots, Y_r;\ W_{r+1}, \cdots, W_n)^T.$$

- $r = \operatorname{rank}(\mathbf{A}) \geqslant 1$. 令$\vec{\alpha}_i = (a_{i1}, \cdots, a_{in})$.
  不妨设前r 行$\vec{\alpha}_1, \cdots, \vec{\alpha}_r$ 线性无关, 且

$$\vec{\alpha}_k = b_{k1}\vec{\alpha}_1 + \cdots + b_{kr}\vec{\alpha}_r, \quad k = r+1, \cdots, n.$$

- $\vec{W} = \mathbf{B}_{(n-r)\times r}\vec{Y}$: $\quad Y_i = \vec{\alpha}_i \cdot \vec{Z}$, $i \leqslant r$,

$$W_k = \vec{\alpha}_k \cdot \vec{Z} = b_{k1}Y_1 + \cdots + b_{kr}Y_r, \quad k = r+1, \cdots, n.$$

- $X_k \overset{\text{a.s.}}{=} b_{k1}X_1 + \cdots + b_{kr}X_r, \quad k = r+1, \cdots, n.$

$$X_k - (b_{k1}X_1 + \cdots + b_{kr}X_r) \overset{d}{=} W_k - (b_{k1}Y_1 + \cdots + b_{kr}Y_r) = 0.$$

- $(X_1, \cdots, X_r)^T \overset{d}{=} \vec{Y} = (Y_1, \cdots, Y_r)^T = \hat{\mathbf{A}}\vec{Z}$ 服从正态分布:

因为$\vec{\alpha}_1, \cdots, \vec{\alpha}_r$ 线性无关, 所以

$$\hat{\mathbf{\Sigma}}_{r\times r} = \hat{\mathbf{A}}\hat{\mathbf{A}}^T = \mathbf{\Sigma}_{11} \quad \text{满秩}, \quad \hat{\mathbf{A}}_{r\times n} = (a_{ij})_{1\leqslant i\leqslant r, 1\leqslant j\leqslant n}.$$

- 利用$\mathbf{\Sigma}$ 非退化情形的结果知,

$$(X_1, \cdots, X_r)^T = \mathbf{C}_{r\times r}\vec{V}, \quad \text{其中}, \vec{V} \sim N(\vec{0}, \mathbf{I}_{r\times r}).$$

- 结论: $\vec{V} = (V_1, \cdots, V_r; V_{r+1}, \cdots, V_n)^T \sim N(\vec{0}, \mathbf{I}_{n\times n})$

$$\vec{X} = \mathbf{M}\vec{V}, \quad \text{其中}, \ \mathbf{M} = \begin{pmatrix} \mathbf{C} & \vec{0} \\ \mathbf{BC} & \vec{0} \end{pmatrix}.$$

- 假设$\vec{X} = (Y_1, \cdots, Y_r; W_{r+1}, \cdots, W_n)^T \sim N(\vec{\mu}, \mathbf{\Sigma})$.

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}.$$

- 假设$\vec{Y}$ 服从正态分布, 即$\mathbf{\Sigma}_{11}$ 非退化. 求$\mathcal{L}(\vec{W}|\vec{Y} = \vec{y})$.
- 不妨假设$E\vec{X} = 0$, 否则考虑$\vec{X} - E\vec{X}$.
- 目标: 找$\mathbf{B}_{(n-r)\times r}$ 使得$\vec{V} = (\vec{W} - \mathbf{B}\vec{Y})$ 与$\vec{Y}$ 不相关, 即

$$\mathrm{cov}(Y_i, V_k) = EY_iV_k = 0, \quad i \leqslant r < k.$$

- 解释: $(Y_1, \cdots, Y_r; V_{r+1}, \cdots, V_n)^T$ 服从高斯分布, 故⋆⋆ 表明$\vec{Y}$ 与$\vec{V}$ 相互独立.

- 协方差: $i \leqslant r < k$,

$$\begin{aligned}\mathrm{cov}(V_k, Y_i) =& E\left(W_k - \sum_{j\leqslant r} b_{kj} Y_j\right) Y_i \\ =& \sigma_{ki} - \sum_{j\leqslant r} b_{kj}\sigma_{ji} = (\mathbf{\Sigma}_{21} - \mathbf{B}\mathbf{\Sigma}_{11})_{ki}.\end{aligned}$$

- 取$\mathbf{B} = \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}$ 即可. 于是$\vec{Y}$ 与$\vec{V}$ 独立, 且$\vec{W} = \mathbf{B}\vec{Y} + \vec{V}$.
- $\vec{V} = \vec{W} - \mathbf{B}\vec{Y} \sim N(\vec{0}, \tilde{\mathbf{\Sigma}}_{22})$, 其中

$$EV_kV_\ell = E\left(W_k - \sum_{j\leqslant r} b_{kj} Y_j\right) V_\ell = EW_k\left(W_\ell - \sum_{j\leqslant r} b_{\ell j} Y_j\right)$$

$$\Rightarrow \tilde{\mathbf{\Sigma}}_{22} = \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}.$$

- 结论: 在$\vec{Y} = \vec{y}$ 的条件下, $\vec{W} = \mathbf{B}\vec{y} + \vec{V}$, 故

$$\mathcal{L}(\vec{W}|\vec{Y} = \vec{y}) = N(\mathbf{B}\vec{y}, \tilde{\mathbf{\Sigma}}_{22}).$$