§7.3 估计的无偏性

- 定义3.1. 若统计量$T = T(X_1, \cdots, X_n)$ 满足

$$E_\theta T = g(\theta), \quad \forall \theta \in \Theta.$$

则称T 为$g(\theta)$ 的无偏估计.

- 例3.1. 样本均值$\bar{X}$ 是期望μ 的无偏估计.

$$\begin{aligned} E_\theta \bar{X} =& E_\theta \frac{1}{n}(X_1 + \cdots + X_n) \\ =& \frac{1}{n}(E_\theta X_1 + \cdots + E_\theta X_n) = \mu. \end{aligned}$$

例3.1(续). 方差σ^2 的无偏估计.

- 样本方差$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ 不是σ^2 的无偏估计.
- $x_i - \mu = (x_i - \bar{x}) + (\bar{x} - \mu)$:

$$\frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 + (\bar{x} - \mu)^2.$$

- $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 - (\bar{X} - \mu)^2$.
- $E_\theta \widehat{\sigma^2} = \text{var}(X) - \text{var}(\bar{X}) = \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$.
- 定理3.1. 若总体方差σ^2 存在, 则S^2 是σ^2 的无偏估计, 其中,

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2.$$

例3.2. 总体: $X \sim \mathrm{Exp}(\lambda)$. 样本量: n. 寻找λ 的无偏估计.

- 最大似然估计& 矩估计: $\hat{\lambda} = 1/\bar{X} = \frac{n}{S_n}$, 其中,

$$S_n = X_1 + \cdots + X_n \sim \Gamma(n, \lambda), \quad p_{S_n}(y) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y},\ y > 0.$$

- 于是,

$$\begin{aligned} E\hat{\lambda} &= E\frac{n}{S_n} = n\int_0^\infty \frac{1}{y} \cdot \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} dy \\ &= n\frac{\lambda\Gamma(n-1)}{\Gamma(n)} \int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\lambda y} dy = \frac{n}{n-1}\lambda. \end{aligned}$$

- $n \geqslant 2$ 时, $\frac{n-1}{n}\hat{\lambda}$ 为λ 的无偏估计.
- $n = 1$ 时, $E\hat{\lambda} = \int_0^\infty \frac{1}{x} \times \lambda e^{-\lambda x} dx = \infty$.