§2.3 连续型随机变量

- 定义3.1. 连续型随机变量指: 存在$p(x)$ 使得

$$P(a \leqslant X \leqslant b) = \int_a^b p(x)dx, \quad \forall a < b.$$

称$p(\cdot)$ 为X 的概率密度(函数), 也记为$p_X(\cdot)$.

- 非负: $p(x) \geqslant 0$; 　规范: $\int_{-\infty}^{\infty} p(x)dx = 1$.
- $P(X = x) = 0$ vs $p(x) \geqslant 0$.
- $p(\cdot)$ 在x 连续, 则$P(X \in [x, x + \Delta x]) = p(x)\Delta x + o(\Delta x)$,
- 单独谈论一个点x 对应的$p(x)$ 没有意义.

1. 均匀分布, $X \sim U(a,b)$ (参数$a < b$):

$$p(x) = \begin{cases} \frac{1}{b-a}; & 若 a \leqslant x \leqslant b; \\ 0, & 否则. \end{cases}$$

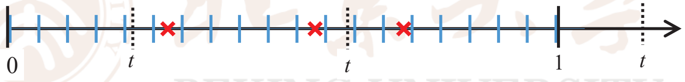
- $a \leqslant b \leqslant x$ 可改为$a < x < b$, $a < x \leqslant b$, $a \leqslant x < b$.
- $p(x) = \frac{1}{b-a} 1_{\{a \leqslant x \leqslant b\}}$.
- $p(x) = \frac{1}{b-a}$, 其中$a \leqslant x \leqslant b$.
- 模型: 几何概型.

2. 指数分布, $X \sim \text{Exp}(\lambda)$ (参数$\lambda > 0$):

$$p(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

- 模型: 例2.3. X = 第一个粒子的放射时刻. 等待时间、寿命.
- 第一个粒子在第Y 个微观时间放出, 则

$$Y \sim G(p), \text{其中} p = \lambda \times \frac{1}{n}.$$



- $X \approx \frac{Y}{n}$, 于是,

$$P(X > t) \approx P(Y > nt) \approx (1-p)^{nt} = (1 - \frac{\lambda}{n})^{nt} \approx e^{-\lambda t}.$$

- $P(X > t) = e^{-\lambda t} = \int_t^{\infty} \lambda e^{-\lambda x} dx.$
- 定理3.1.(无记忆性): $P(X - s > t | X > s) = e^{-\lambda t}, \forall t, s \geqslant 0.$

3. 正态分布, $X \sim N(\mu, \sigma^2)$ (参数$\mu \in \mathbb{R}$, $\sigma > 0$):

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

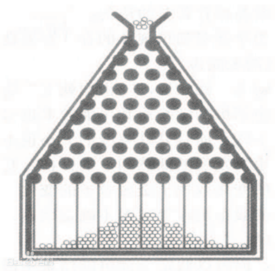
- 标准正态分布$N(0,1)$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

- §1.7. 第二近似公式: $x_k = \frac{k-np}{\sqrt{npq}}$,

$$C_n^k p^k q^{n-k} \approx \frac{1}{\sqrt{npq}} \phi(x_k).$$

高尔顿钉板试验



- $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, 将积分的平方写为二重积分:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dxdy.$$

- 做极坐标变换:

$$x = r\cos\theta,\ y = r\sin\theta \Rightarrow \left| \begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{matrix} \right| = r.$$

- 因此,

$$\star\star = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} e^{-\frac{r^2}{2}} r dr \right) d\theta = \int_0^{\infty} e^{-R} dR = 1.$$

- $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$: 令$y = \frac{x-\mu}{\sigma}$, 则

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1.$$

- 函数Φ:

$$\Phi(x) = \int_{-\infty}^{x} \phi(x) dx.$$

- $\Phi(-x) = 1 = \Phi(x)$.
- 定理3.2. 令$x^* = \frac{x-\mu}{\sigma}$, 则

$$P(a < X < b) = \int_a^b \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) dx = \Phi(b^*) - \Phi(a^*).$$

- 推论3.1. 查表得$\Phi(3) = 0.9987$, 因此

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = \Phi(3) - \Phi(-3) = 0.9974.$$

4. 威布尔(Weibull)分布, $X \sim \mathrm{W}(m, \eta)$ (参数$m, \eta > 0$):

$$p(x) = \frac{m}{\eta^m} x^{m-1} \exp\left\{ - \left(\frac{x}{\eta} \right)^m \right\}, \quad x > 0.$$

- $\int_0^\infty m y^{m-1} e^{-y^m} dy = \int_0^\infty e^{-z} dz = 1.$
- $m > 0$: 形状参数; $\eta > 0$: 尺度参数.
- $m = 1$ 时就是指数分布$\mathrm{Exp}(\frac{1}{\eta})$.
- 应用: 机电产品的寿命; 可靠性研究.

5. 伽玛分布, $X \sim \Gamma(\alpha,\beta)$ (参数$\alpha,\beta>0$):

$$p(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x},\quad x>0.$$

- $\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy$.
- $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$:

$$\int_0^\infty y^\alpha e^{-y}dy=-y^\alpha e^{-y}|_0^\infty+\int_0^\infty \alpha y^{\alpha-1}e^{-y}dy.$$

- $\Gamma(1)=1$; $\Gamma(\frac{1}{2})=\sqrt{\pi}$:

$$\Gamma(\tfrac{1}{2})=\int_0^\infty \frac{1}{\sqrt{y}}e^{-y}dy=\sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}}dx=\sqrt{\pi}.$$

- $\alpha=1$ 时就是指数分布Exp(β).