Conductor formulas for motivic spectra

Introduction

Joint with Fangzhou Jin

Beilinson's philosopy in 2007

k : perfect field of Char p

$\Lambda = \mathbb{F}_\ell, \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ with $\ell \neq p$.

$X_{/k}$: proper smooth scheme of dimension d

We consider the trace and determinant of $R\Gamma(X, \mathcal{F})$ for motivic cohomology $R\Gamma(X, -)$.

For étale cohomology, we define

trace = Euler-Poincaré characteristic number

$$\chi(X, \mathcal{F}) = \mathrm{tr}(\mathrm{id}; R\Gamma(X_{\bar{k}}, \mathcal{F}))$$
$$= \sum (-1)^i \dim H^i(X_{\bar{k}}, \mathcal{F})$$

determinant = global epsilon line / factor

$$\varepsilon(X, \mathcal{F}) = \det R\Gamma(X_{\bar{k}}, \mathcal{F})^{-1}$$
$$= \bigotimes_i \left(\det H^i(X_{\bar{k}}, \mathcal{F})\right)^{(-1)^{i+1}}$$

If $k=\mathbb{F}_p$ finite, $L(X,\mathcal{F},t)=\det(1-t\cdot \mathrm{Frob}_k; R\Gamma(X_{\bar{k}},\mathcal{F}))^{-1}$
Grothendieck L-function

Functional equation
$$L(X,\mathcal{F},t)=t^{\chi(X,\mathcal{F})}\cdot \varepsilon(X,\mathcal{F})\cdot L(X,D(\mathcal{F}),t^{-1})$$
$$\varepsilon(X,\mathcal{F})=\det(-\mathrm{Frob}_k; R\Gamma(X_{\bar{k}},\mathcal{F}))^{-1}$$

- When X is a projective smooth curve, Laumon proved that
$$\varepsilon(X,\mathcal{F})=\prod_{x\in|X|}\varepsilon_x(X,\mathcal{F},\omega)\ ,\quad \omega\in\Omega^1_{k(X)}\setminus\{0\}$$
local epsion factor/Line.

which is conjectured by Langlands and Deligne.

- When X is a projective smooth curve, we have Grothendieck Ogg-Safarevich formula
$$\chi(X,\mathcal{F})=-\sum_{x\in|X|}\deg(x)\left(\dim\mathcal{F}_{\bar{\eta}_x}\cdot \mathrm{ord}_x(\omega)+\mathrm{art}_x(X,\mathcal{F})\right)$$
$$=\mathrm{rank}\,\mathcal{F}\cdot\chi(X,\Lambda)-\sum_{x\in|X|}\mathrm{art}_x(X,\mathcal{F})$$

where $\mathrm{art}_x(X,\mathcal{F})=\dim\mathcal{F}_{\bar{\eta}_x}+\mathrm{Sw}_x(\mathcal{F})-\dim\mathcal{F}_{\bar{x}}$ is the Artin conductor, $\mathrm{Sw}_x(\mathcal{F})$ is the Swan conductor, measuring the wild ramification of $\mathcal{F}$ around x.

Higher dimensional analogues?

Open Question A (after Beilinson and Saito)

(1) There should exists a K-theory spectrum $\overset{?}{K}(X,\Lambda)$, for constructible motives on X.
an $\mathbb{E}_\infty$-ring

(2) For any constructible motive $\mathcal{F}$, there exists a d-cycle $\mathcal{E}(\mathcal{F})$ on T^*X with coefficients in the $\mathbb{E}_\infty$-ring $\overset{?}{K}(k,\Lambda)$ such that the following Dubson~Kashiwara style formula holds:

$$(*) \quad [R\Gamma(X_{\bar{k}},\mathcal{F})] = \langle \mathcal{E}(\mathcal{F}), X\rangle_{T^*X} \simeq \bigoplus_v \mathcal{E}_v(X,\mathcal{F})$$

as homotopy points of $\overset{?}{K}(k,\Lambda)$.

Take trace of id in $(*)$, we get index formula

$$\chi(X,\mathcal{F}) = \sum_v tr(id, \mathcal{E}_v(X,\mathcal{F}))$$

Take determinant, get product formula for global epsilon factor

$$\mathcal{E}(X,\mathcal{F}) = \bigotimes_v \det \mathcal{E}_v(X,\mathcal{F})$$

where $\det \mathcal{E}_v(X\ \mathcal{F})$ should be the local epsilon factor/line.

In order to study pull-back, we have to construct a relative version

(3) <u>Relative Version</u> for X/S under ULA condition.

In order to prove Milnor-type formula for non-smooth objects, we need to define the $\mathcal{E}$-version of non-acyclicity class

(4) <u>Non-acyclicity Class</u> for $(X/S, \mathcal{F})$ [Joint with Yigeng Zhao]

(5) <u>$\mathcal{E}$-Conductor formula</u> (Proper Push-forward)

Consider a proper morphism $f: X \to Y$ between smooth schemes$/k$ satisfying certain transversal conditions w.r.t $\mathcal{F} \in K^{?}(X, \Lambda)$, we have

$$f_* \mathcal{E}(\mathcal{F}) = \mathcal{E}(f_* \mathcal{F})$$ as cycles with coefficients in $K^{?}(Y, \Lambda)$

Put $cc_X(\mathcal{F}) = \langle \mathcal{E}(\mathcal{F}), X \rangle_{T^*X}$

$$f_* cc_X(\mathcal{F}) = cc_X(f_* \mathcal{F})$$

"$\mathcal{E}$-characteristic class"

(6) <u>$\mathcal{E}$-Milnor formula</u>

$X \xrightarrow{f} C$ fibration to a smooth curve C/k.

$\mathcal{F} \in K^{?}(X, \Lambda)$.

$x \in |X|$ isolated singularity w.r.t $(X \xrightarrow{f} C, \mathcal{F})$

Then we expect

$$(**) \quad \left[\mathcal{E}_{f(x)}(R\Phi_{\bar{x}}(X,\mathcal{F}))\right] = \langle \mathcal{E}(\mathcal{F}), df \rangle_{T^*X,x}$$

as homotopy points of $K^{?}(G_{f(x)},\Lambda)$, where $R\Phi$ is the vanishing cycle functor.

If take trace, one get Classical Milnor formula and local epsilon factor of vanishing cycles.

(For $\mathcal{F}=\Lambda$, see Daichi Takeuchi)

<u>As a corollary of (**)</u>

$\mathcal{E}(\mathcal{F})$ is supported on the singular support $SS\mathcal{F} \subseteq T^*X$

$tr(id, \mathcal{E}(\mathcal{F}))$ is the characteristic cycle of $\mathcal{F}$.

(7) <u>Milnor formula for non-isolated singularities</u> (Need to use non-acyclicity class)

<u>Some known results</u>

Beilinson 2007 for Betti cohomology

Patel 2012 for $\mathcal{D}_X$-modules.

Beilinson + Saito : for $K_0(X,\Lambda)$ $\begin{cases} \text{singular support} \\ \text{characteristic cycle.} \end{cases}$

Quentin Guignard 2020 : numerical solution for the product formula for higher $\mathcal{E}$-factor

Joint with Fangzhou Jin, we obtained a quadratic version.

(take $K^? = GW$)

<u>Qudratic invariants</u>

People usually apply "$\mathbb{A}^1$-homotopy" to refine $\mathbb{Z}$-valued enumerative geometry ~~to~~ with values in quadratic forms.

$$\begin{tikzcd}
\text{Classical invariants} \arrow[rr, Rightarrow, "\mathbb{A}^1\text{-homotopy}"] & & \text{Quadratic refinements} \arrow[ll, bend right=50, "{-/\mathbb{R}\quad \text{signature (related to } \mathbb{R}\text{-topo)}}"'] \arrow[ll, bend left=50, "\text{rank}"]
\end{tikzcd}$$

<u>Foundation</u> : Barge, Morel, Fasel.

<u>Application</u> : Hoyois (quadratic Groth-Lef-Verdier trace formula)

Kass-Wickelgren : $\mathbb{A}^1$-Milnor formula

M. Levin : quadratic Euler-Poincaré char number.

Our approach to quadratic conductor formula (Open question A) is based on $\begin{cases} \text{quadratic refinements of Artin conductors} \\ \text{quadratic refinements of GOS formula} \end{cases}$

§1 The Grothendieck-Witt ring

k : any field.

$GW(k)$ = group completion of the semi-ring

$$\left\{ \begin{array}{c} \text{non-degenerate} \\ \text{symmetric bilinear} \\ \text{forms} / k \end{array} \right\}^{\otimes} / \cong$$

If char $k \neq 2$, this is equiv to non-deg quadratic forms.

$$\mathrm{rk} = \mathrm{rank} : GW(k) \longrightarrow \mathbb{Z}$$
$$A \longmapsto \dim A$$

If $k = \overline{k}$ is alg. closed, then $GW(k) \overset{\mathrm{rk}}{\cong} \mathbb{Z}$.

Given $u \in k^\times$, let $\langle u \rangle : k \times k \longrightarrow k$ be the sym. bilinear form $\langle u \rangle (a, b) = uab$.

Then $GW(k)$ is a ring generated by $\langle u \rangle$, $u \in k^\times/(k^\times)^2$, subj to the following relations:

- $\langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$
- $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$ if $a+b \neq 0$
- $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$

$H := \langle 1 \rangle + \langle -1 \rangle$, called the hyperbolic element of $GW(k)$.

Example (1) $GW(\mathbb{C}) \cong \mathbb{Z}$.

(2)
$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r, "(rk, sig)", "\text{u}"'] \arrow[d, "\cong", "\text{ring isom}"'] & \mathbb{Z}\times\mathbb{Z} \\ \mathbb{Z}[t]/(t^2-1) & \end{tikzcd}$$
(not ring isom)

$\langle 1\rangle \mapsto 1$, $\langle -1\rangle \mapsto t$

(3) $GW(\mathbb{F}_q) \xrightarrow[\text{u}]{(rk, det)} \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$

(4) K : Complete DVF with residue field F, char $F \neq 2$, then $GW(K) \cong \dfrac{GW(F) \oplus GW(F)}{\mathbb{Z}[H, -H]}$

Morel's isomorphism

$$GW(k) \cong \mathrm{End}_{SH(k)}(\mathbb{1}_k)$$

$SH(k)$: Stable motivic homotopy category.

$\mathbb{1}_k \in SH(k)$ motivic sphere spectrum.

Sm/k : Cat of smooth schemes$/k$.

$$\begin{tikzcd} Sm_{/k} \arrow[r, "\Sigma^\infty_+"] \arrow[dr, "C^\ell_*"'] & SH(k) \arrow[d, "\mathbb{R}_\ell"] \\ & \hat{D}(\mathrm{Spec}\, k_{\acute{e}t}, \mathbb{Z}_\ell) \end{tikzcd}$$

Rough idea to do quadratic refinements:

Find a categorical approach to your invariants, then do it in $SH(-)$.

<u>Recall</u> $SH(S)$ = limit of presentable ∞-cat of

$$\cdots \xrightarrow{\Omega} \mathcal{H}_*(S) \xrightarrow{\Omega} \mathcal{H}_*(S)$$

where $\Omega(Z) = \underline{\mathrm{Hom}}(\mathbb{P}^1, Z)$ is the ∞-loop functor.

The $\mathbb{A}^1$-homotopy (or motivic homotopy) ∞-Cat $\mathcal{H}_*(S)$ of spaces over S is the ∞-cat of functors $\mathcal{F}: (Sm/S)^{op} \longrightarrow \mathcal{S}_*$ (∞-cat of pointed spaces) satisfying:

— $\mathbb{A}^1$-invariance $\forall X \in Sm_S$, $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1_X)$ is a weak equivalence in $\mathcal{S}_*$.

— Excision. $\forall X, Y \in Sm/S$, $\forall$ excisive morphism

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

$\mathcal{F}(X,Z) :=$ homotopy fiber of $\mathcal{F}(X) \to \mathcal{F}(X \setminus Z)$

The map $p_*: \mathcal{F}(Y,T) \to \mathcal{F}(X,Z)$ is a weak equivalence in $\mathcal{S}_*$.

<u>Basic examples of quadratic refinements of invariants</u>

If X/k proper smooth, then $X = \Sigma^+ X$ is strongly dualisable in the symmetric monoidal category $SH(k)$, we write its dual by $X^\vee$.

Then $\chi(X/k) = (\mathbb{1}_k \xrightarrow{\text{unit}} \mathrm{Hom}(X,X) \cong X \otimes X^\vee \xrightarrow[\text{evaluation}]{\text{counit}} \mathbb{1}_k)$

in $\mathrm{End}_{SH(k)}(\mathbb{1}_k) = GW(k)$.

Which is called the cat. Euler-Char.

<u>Serre's remark</u>: this symmetric bilinear form is given by the Poincaré duality.

$$\chi(\mathbb{P}^n/k) = \begin{cases} \frac{n+1}{2} \mathbb{H} & n \text{ odd} \\ \langle 1 \rangle + \frac{n}{2} \cdot \mathbb{H} & n \text{ even} \end{cases}$$

$$\left(\begin{array}{l} \text{For } k = \mathbb{R} \subseteq \mathbb{C}, \\ \chi(X/\mathbb{R}) \in GW(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z} \\ \begin{tikzcd} & \chi(X/\mathbb{R}) \arrow[dl, "\text{rank}"'] \arrow[r, "\text{sig}"] & \chi(X(\mathbb{R})) \\ \chi(X(\mathbb{C})) & & \end{tikzcd} \end{array} \right)$$

Theorem 1 (Jin–Y) Quadratic refinement of GOS formula

$\mathrm{char}(k) \neq 2$ and $\dim X$ odd

$$\begin{tikzcd} Z \arrow[r, hook] & X \arrow[d, "p: \text{proper smooth.}"] \\ & \mathrm{Spec}\, k \end{tikzcd}$$

$K \in SH_c(X)$ such that $K|_{X \setminus Z}$ is dualizable, then

Quadratic artin conductor

$$\underset{\substack{\| \\ \chi(p_* K)}}{\chi(X, K)} = \underset{\substack{| \\ \mathrm{rk}(\text{et relalize of } K)}}{\mathrm{rk}\, K} \cdot \chi(X/k) - \mathrm{Art}(K) \quad \text{in } GW(k)$$

When X is a curve, Z: finite set of closed points, taking rank + étale realization, get

$$\mathrm{rk}(\mathrm{Art}(K)) = \sum_{x \in Z} a_x(R_\ell(K))$$

Artin conductor of $R_\ell(K)$ at $x \in Z$

$$a_x(\mathcal{F}) = \mathrm{rank}\, \mathcal{F}_{\eta_x} - \mathrm{rank}\, \mathcal{F}_{\bar{x}} + \mathrm{Sw}_x \mathcal{F}.$$

Theorem 2 (Jin–Y) Quadratic Bloch conductor formula.

$$\begin{array}{l} X \\ \big\downarrow f: \text{proper, flat, smooth outside } X_y. \\ y \in Y/k \text{ proper smooth curve} \end{array}$$

Then we have

$$-\underset{\substack{\uparrow \\ \text{quadratic Artin}}}{\mathrm{Art}}(Rf_*\hat{F}) = f_!\left(\begin{array}{c}\text{a class in} \\ H_{MW}\Lambda(X_y/y)\end{array}\right) \text{ in } GW(k(y)) \to GW(k)$$

$H_{MW} \to GW(k)$, Λ: Milnor–Witt spectrum

Which generalize Bloch's conductor formula (conj 1987)

$$-a_y(Rf_*\Lambda) = (-1)^{d+1}\deg\left(C^X_{d+1,X_y}(\Omega^1_{X/Y})\cap[X]\right)$$

d = rel. dim of $X \to Y$.

proved under
- $d=1$ Bloch
- higher dim 2020 by T. Saito
- $(X_y)_{red}$ NCD, Kato–Saito
- reproved by Y–Zhao, 2022.

§2 Non-acyclicity Class

Notation $\Delta = \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & & S \end{tikzcd} \right)$

S: Noetherian scheme
g: smooth

in $Sch_{/S} = \left\{ \begin{array}{c} \text{sep of f.t} \\ \text{over } S \end{array} \right\}$

Call $Z =$ NA locus of Δ

$$D_c^b(\Delta) = \left\{ \mathcal{F} \in D_c^b(X, \Lambda) \;\middle|\; \begin{array}{l} h \text{ is } \mathcal{F}\text{-ULA} \\ f \text{ is } \mathcal{F}\text{-ULA outside } Z \end{array} \right\}$$

Groth. 6-functor formalism

$$\begin{tikzcd} \mathrm{Corr}_S \arrow[r] & \mathrm{Pres} \\ \mathcal{C}_S \arrow[u] \end{tikzcd}$$

(Un) straighten

2-CoCartesian fibration

$(X, \mathcal{F}) \in \mathcal{C}_S$ is rigid/dualisable

$\Updownarrow$

$X \to S$ is $\mathcal{F}$-ULA.

For any $\mathcal{F} \in D_c^b(\Delta)$, we will define

$$\mathcal{K}_\Delta \in D_c^b(X, \Lambda)$$

a class $C_\Delta(\mathcal{F}) \in H_Z^0(X, \mathcal{K}_\Delta) \; \left(\cong H^0(Z, \mathcal{K}_{Z/S}) \right)$

when Z is small enough,

i.e, $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0.$

Thm (Y-Zhao, 2022)

(1) Given $S' \xrightarrow{b} S$, get Δ' by base change, then

$$b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(\mathcal{F}')$$

(2) Given $\Delta \xrightarrow{\text{proper}} \Delta'$, in the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & Z' \arrow[d, hook] \\ X \arrow[r, "s"] \arrow[dr] \arrow[ddr, bend right] & X' \arrow[d] \arrow[dd, bend left] \\ & Y \arrow[d] \\ & S \end{tikzcd}$$

s: proper

then $s_* C_\Delta(\mathcal{F}) = C_{\Delta'}(s_*\mathcal{F})$

(3) cohomological Milnor formula

Take $\Delta = \left(\begin{tikzcd} Z = \{x\} \arrow[r, hook] & X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & & \operatorname{Spec} k \end{tikzcd} \right)$ $\quad Y$ smooth curve

Then $C_\Delta(\mathcal{F}) = -\operatorname{dimtot} R\Phi_x(\mathcal{F}, f)$, $\mathcal{F} \in D_c^b(\Delta)$.

$\cap$

$H^0(\{x\}, \Lambda) = \Lambda \cdot \{x\}$

(4) cohomological conductor formula

If $X \xrightarrow{f} Y$ proper, Y: curve, $y \in |Y|$, $Z = f^{-1}(y)$

$$\Delta = \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & & \operatorname{Spec} k \end{tikzcd} \right) \xrightarrow{f} \Delta' = \left(\begin{tikzcd} \{y\} \arrow[r, hook] & Y \arrow[r, equal] \arrow[dr] & Y \arrow[d] \\ & & \operatorname{Spec} k \end{tikzcd} \right)$$

Apply (2) to $\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & \{y\} \arrow[d, hook] \\ X \arrow[r] \arrow[dr] & Y \arrow[d, equal] \\ & Y_{/k} \end{tikzcd}$ get $f_* C_\Delta(\mathcal{F}) = C_{\Delta'}(f_*\mathcal{F}) \overset{\text{Milnor}}{=\!=} -\operatorname{dimtot} R\Phi_y(f_*\mathcal{F}, \mathrm{id})$

$= -a_y(Rf_*\mathcal{F})$

(5) Cohomological GOS formula (char $k = p > 0$)

$$\Delta = \left(\begin{array}{c} Z \hookrightarrow X = X \\ \searrow \ \downarrow \\ \mathrm{Spec}\, k \end{array} \right), \quad X : \text{smooth curve}$$

$$\mathcal{F} \in D_c^b(\Delta) \ (\Leftrightarrow \mathcal{F}|_{X \setminus Z} \text{ smooth})$$

$$C_\Delta(\mathcal{F}) = -\sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \text{ in } H^0(Z, K_{Z/k})$$

Construction of $C_\Delta(\mathcal{F})$

$$\begin{tikzcd} X' = X \arrow[r, "i"] \arrow[d, "\delta_1"] \arrow[dd, bend right, "f"'] & X \arrow[d, "\delta_0"] \\ X \times_Y X \arrow[r, "i"] \arrow[d, "p"] & X \times_S X \arrow[d, "f \times f"] \\ Y \arrow[r, "\delta"] & Y \times_S Y \end{tikzcd}$$

For $\mathcal{F} \in D_c^b(\Delta)$, put

$$C_{X/S}(\mathcal{F}) = \left(\delta_{0!}\Lambda = \delta_{0*}\Lambda \to \mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F}) \to \delta_{0*} K_{X/S} \right)$$

We define a functor $\delta^\Delta : D_c^b(X) \longrightarrow D_c^b(X')$

$$i^* K \otimes f^* \delta^! \Lambda \to i^! K \to \delta^\Delta K \xrightarrow{+1}$$

Technical Lemma $\delta^\Delta C_{X/S}(\mathcal{F})$ is supported on Z

Then we define $C_\Delta(\mathcal{F}) = \delta^\Delta C_{X/S}(\mathcal{F})$.

The most difficult part is to prove the following conjecture, formulated by myself with Zhao.

<u>Conjecture</u> If $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$, then

$$C_{X/S}(\mathcal{F}) = C_r(f^*\Omega^{1,\vee}_{Y/S}) \cap C_{Y/S}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in } H^0(X, \mathcal{K}_{X/S}).$$

Y-Zhao: true if $Z = \phi$

Abbes-Saito: If $f = id$, and $S = \operatorname{Spec} k$, OK.

Y-Zhao : $S = \operatorname{Spec} k$, Y: smooth curve, Z: finite set of closed points.

$$C_{X/S}(\mathcal{F}) = C_1(f^*\Omega^{1,\vee}_{Y/k}) \cap C_{Y/S}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \begin{bmatrix}\text{fibration}\\ \text{formula}\end{bmatrix}$$

$$C_\Delta(\mathcal{F}) = -\sum_{x \in Z} \operatorname{dimtot} R\Phi(\mathcal{F}, f) \cdot [x].$$

<u>More conjecture</u> (mixed version)

$$\blacksquare = \begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ T & \xhookrightarrow{\delta} & S \end{array} \qquad D^b_c(\blacksquare) = \left\{ \mathcal{F} \in D^b_c(X, \Lambda) \;\middle|\; \begin{array}{c} X \setminus Z \to S \\ \text{is } \mathcal{F}\text{-ULA} \end{array} \right\}$$

δ is a regular embedding. [是否可去掉 T]

Then $\exists\, \mathcal{K}_{\blacksquare}$ and $\exists$ class $C_{\blacksquare}(\mathcal{F}) \in H^0_Z(X, \mathcal{K}_{\blacksquare})$ with similar properties.