

# Conductor formulas for motivic spectra

## Introduction

Joint with Fangzhou Jin

Beilinson's philosophy in 2007

$k$ : perfect field of char  $p$

$\Lambda = \mathbb{F}_\ell, \mathbb{Z}_\ell$  or  $\mathbb{Q}_\ell$  with  $\ell \neq p$ .

$X/k$ : proper smooth scheme of dimension  $d$

We consider the trace and determinant of  $R\Gamma(X, \mathcal{F})$  for motivic cohomology  $R\Gamma(X, -)$ .

For étale cohomology, we define

trace = Euler-Poincaré characteristic number

$$\chi(X, \mathcal{F}) = \text{tr}(\text{id}; R\Gamma(X_{\bar{k}}, \mathcal{F}))$$

$$= \sum (-1)^i \dim H^i(X_{\bar{k}}, \mathcal{F})$$

determinant = global epsilon line / factor

$$\begin{aligned} \mathcal{E}(X, \mathcal{F}) &= \det R\Gamma(X_{\bar{k}}, \mathcal{F})^{-1} \\ &= \bigotimes_i (\det H^i(X_{\bar{k}}, \mathcal{F}))^{(-1)^{i+1}} \end{aligned}$$

If  $k = \mathbb{F}_p$  finite,  $L(X, \mathcal{F}, t) = \det(1 - t \cdot \text{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$   
 Grothendieck L-function

Functional equation

$$L(X, \mathcal{F}, t) = t^{\chi(X, \mathcal{F})} \cdot \varepsilon(X, \mathcal{F}) \cdot L(X, D(\mathcal{F}), t^{-1})$$

$$\varepsilon(X, \mathcal{F}) = \det(-\text{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$$

► When  $X$  is a projective smooth curve, Laumon proved that

$$\varepsilon(X, \mathcal{F}) = \prod_{x \in |X|} \varepsilon_x(X, \mathcal{F}, \omega), \quad \omega \in \Omega_{k(x)}^1 \setminus \{0\}$$

↑ local epsilon factor / Lme.

which is conjectured by Langlands and Deligne.

► When  $X$  is a projective smooth curve, we have Grothendieck Ogg-Safarevich formula

$$\chi(X, \mathcal{F}) = - \sum_{x \in |X|} \deg(x) \left( \dim \hat{\mathcal{F}}_{\eta_x} \cdot \text{ord}_x(\omega) + \text{art}_x(X, \mathcal{F}) \right)$$

$$= \text{rank } \mathcal{F} \cdot \chi(X, \mathcal{N}) - \sum_{x \in |X|} \text{art}_x(X, \mathcal{F})$$

where  $\text{art}_x(X, \mathcal{F}) = \dim \hat{\mathcal{F}}_{\eta_x} + \text{Sw}_x(\mathcal{F}) - \dim \hat{\mathcal{F}}_{\bar{x}}$  is the Artin conductor,  $\text{Sw}_x(\mathcal{F})$  is the Swan conductor, measuring the wild ramification of  $\mathcal{F}$  around  $x$ .

Higher dimensional analogues?

Open Question A (after Beilinson and Saito)

(1) There should exist a K-theory spectrum  $K^?(X, \Lambda)$ , for an  $\mathbb{E}_\infty$ -ring  
constructible motives on  $X$ .

(2) For any constructible motive  $\mathcal{F}$ , there exists a d-cycle  $\mathcal{E}(\mathcal{F})$  on  $T^*X$  with coefficients in the  $\mathbb{E}_\infty$ -ring  $K^?(k, \Lambda)$  such that the following Dubson-Kashimura style formula holds:

(\*)  $[R\Gamma(X_{\mathbb{R}}, \mathcal{F})] = \langle \mathcal{E}(\mathcal{F}), X \rangle_{T^*X} \simeq \bigoplus_{\nu} \mathcal{E}_{\nu}(X, \mathcal{F})$   
as homotopy points of  $K^?(k, \Lambda)$ .

Take trace of id in (\*), we get index formula

$$\chi(X, \mathcal{F}) = \sum \text{tr}(\text{id}, \mathcal{E}_{\nu}(X, \mathcal{F}))$$

Take determinant, get  $\sqrt{\phantom{x}}$  product formula for global epsilon

factor  $\mathcal{E}(X, \mathcal{F}) = \bigotimes_{\nu} \det \mathcal{E}_{\nu}(X, \mathcal{F})$

where  $\det \mathcal{E}_{\nu}(X, \mathcal{F})$  should be the local epsilon factor/line.

In order to study pull-back, we have to construct a relative version

(3) Relative Version for  $X/S$  under ULA condition.

In order to prove Milnor-type formula for non-smooth objects, we need to define the  $\mathcal{E}$ -version of non-acyclicity class

(4) Non-acyclicity class for  $(X/S, \mathcal{F})$  [Joint with Yigeng zhang]

(5)  $\mathcal{E}$ -Conductor formula (Proper Push-forward)

Consider a proper morphism

$f: X \rightarrow Y$  between smooth schemes/ $k$ , satisfying certain transversal conditions w.r.t  $\mathcal{F} \in K^?(X, \mathcal{N})$ , we have

$$f_* \mathcal{E}(\mathcal{F}) = \mathcal{E}(f_* \mathcal{F}) \quad \text{as cycles with coefficients in } K^?(Y, \mathcal{N})$$

$$\text{Put } cc_X(\mathcal{F}) = \langle \mathcal{E}(\mathcal{F}), X \rangle_{Y^*X}$$

$$\uparrow f_* cc_X(\mathcal{F}) = cc_X(f_* \mathcal{F})$$

" $\mathcal{E}$ -characteristic class"

(6)  $\mathcal{E}$ -Milnor formula

$X \xrightarrow{f} C$  fibration to a smooth curve  $C/k$ .

$\mathcal{F} \in K^?(X, \mathcal{N})$ .

$x \in |X|$  isolated singularity w.r.t  $(X \xrightarrow{f} C, \mathcal{F})$

Then we expect

(\*\*)  $[E_{f, \text{loc}}(R\Phi_{\bar{x}}(X, \mathcal{F}))] = \langle E(\mathcal{F}), df \rangle_{T^*X, x}$   
 as homotopy points of  $K^?(G_{f, \text{loc}}, \Lambda)$ , where  $R\Phi$  is the  
 vanishing cycle functor.

If take trace, one get Classical Milnor formula  
 and local epsilon factor of vanishing cycles.

(For  $\mathcal{F} = \Lambda$ , see Daichi Takeuchi)

As a corollary of (\*\*)

$E(\mathcal{F})$  is supported on the singular support  $SS\mathcal{F} \subseteq T^*X$   
 $\text{tr}(\text{id}, E(\mathcal{F}))$  is the characteristic cycle of  $\mathcal{F}$ .

(7) Milnor formula for non-isolated singularities (Need to use  
 non-acyclicity  
 class)

Some known results

Beilinson 2007 for Betti cohomology

Patel 2012 for  $\mathcal{D}_X$ -modules.

Beilinson + Saito: for  $K_0(X, \Lambda)$   $\left\{ \begin{array}{l} \text{singular support} \\ \text{characteristic cycle.} \end{array} \right.$

Quentin Guignard 2020: numerical solution for the product  
 formula for higher  $E$ -factor

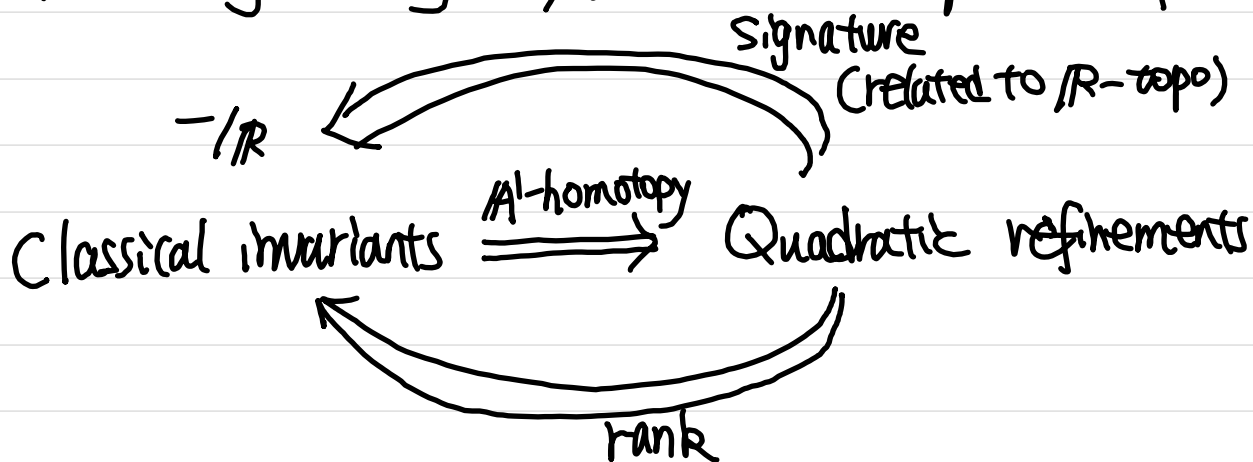
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Joint with Fangzhou Jin, we obtained a quadratic version.

(take  $K^? = GW$ )

### Quadratic invariants

People usually apply " $A^1$ -homotopy" to refine  $\mathbb{Z}$ -valued enumerative geometry to/with values in quadratic forms.



Foundation = Barge, Morel, Fasel.

Application: Hoyois (quadratic Groth-Lef-Verdier trace formula)

Kass-Wickelgren =  $A^1$ -Milnor formula

M. Levine: quadratic Euler-Poincaré char number.

Our approach to quadratic conductor formula (Open question A)

is based on  $\left\{ \begin{array}{l} \text{quadratic refinements of Artin conductors} \\ \text{quadratic refinements of GOS formula} \end{array} \right.$

## § 1 The Grothendieck-Witt ring

$k$ : any field.

$GW(k)$  = group completion of the semi-ring

$$\left\{ \begin{array}{l} \text{non-degenerate} \\ \text{symmetric bilinear} \\ \text{forms / } k \end{array} \right\}^{\otimes} / \cong$$

If  $\text{char } k \neq 2$ ,  
this is equiv  
to non-deg  
quadratic  
forms.

$$\text{rk} = \text{rank} : GW(k) \longrightarrow \mathbb{Z}$$

$$A \longmapsto \dim A$$

If  $k = \mathbb{R}$  is alg. closed, then  $GW(k) \stackrel{\text{rk}}{\cong} \mathbb{Z}$ .

Given  $u \in k^x$ , let  $\langle u \rangle : k \times k \rightarrow k$  be the sym. bilinear form  
form  $\langle u \rangle(a, b) = uab$ .

Then  $GW(k)$  is a ring generated by  $\langle u \rangle$ ,  $u \in k^x / (k^x)^2$ ,  
subj to the following relations:

$$\text{--- } \langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$$

$$\text{--- } \langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle \text{ if } a+b \neq 0$$

$$\text{--- } \langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$$

$H := \langle 1 \rangle + \langle -1 \rangle$ , called the hyperbolic element  
of  $GW(k)$ .

Example (1)  $GW(\mathbb{C}) \cong \mathbb{Z}$ .

$$(2) \quad GW(\mathbb{R}) \xrightarrow[\cong]{(rk, sig)} \mathbb{Z} \times \mathbb{Z} \quad (\text{not ring isom})$$

$$\begin{array}{ccc} & \langle 1 \rangle & \langle -1 \rangle \\ & \downarrow & \downarrow \\ \text{ring isom} \Big| \cong & & \\ \downarrow & & \\ \mathbb{Z}[t]/(t^2-1) & & \end{array}$$

$$(3) \quad GW(\mathbb{F}_q) \xrightarrow[\cong]{(rk, det)} \mathbb{Z} \times \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$$

$$(4) \quad K: \text{Complete DVF with residue field } F, \text{ char } F \neq 2, \\ \text{then } GW(K) \cong \frac{GW(F) \oplus GW(F)}{\mathbb{Z}[H, -H]}$$

Morel's isomorphism

$$GW(k) \cong \text{End}_{SH(k)}(\mathbb{I}_k)$$

$SH(k)$  : stable motivic homotopy category.

$\mathbb{I}_k \in SH(k)$  motivic sphere spectrum.

$Sm/k$  : Cat of smooth schemes/k.



$$\begin{array}{ccc}
 Sm/k & \xrightarrow{\Sigma_+^\infty} & SH(k) \\
 & \searrow \mathcal{C}_*^{\mathcal{L}} & \downarrow \mathbb{R}_2 \\
 & & \hat{D}(\text{Spec } k_{\text{et}}, \mathbb{Z}_\ell)
 \end{array}$$

Rough idea to do quadratic refinements:

Find a categorical approach to your invariants, then do it in  $SH(-)$ .

Recall  $SH(S) = \text{limit of presentable } \infty\text{-cat of}$   
 $\dots \xrightarrow{\Omega} \mathcal{H}_*(S) \xrightarrow{\Omega} \mathcal{H}_*(S)$   
 where  $\Omega(\mathbb{Z}) = \underline{\text{Hom}}(\mathbb{P}^1, \mathbb{Z})$  is the  
 $\infty$ -loop functor.

The  $A^1$ -homotopy (or motivic homotopy)  $\infty$ -cat  $\mathcal{H}_*(S)$  of spaces over  $S$  is the  $\infty$ -cat of functors

$$\mathcal{F} = (Sm/S)^{\text{op}} \longrightarrow \mathcal{L}_* (\infty\text{-cat of pointed spaces}) \text{ satisfying:}$$

—  $A^1$ -invariance  $\forall X \in Sm_S, \mathcal{F}(X) \rightarrow \mathcal{F}(A_X^1)$  is a weak equivalence in  $\mathcal{L}_*$ .

— Excision.  $\forall X, Y \in Sm/S, \forall$  excisive morphism

$$\begin{array}{ccc} T & \hookrightarrow & Y \\ \downarrow & \square & \downarrow p \\ Z & \hookrightarrow & X \end{array}$$

$\mathcal{F}(X, Z) :=$  homotopy fiber of  $\mathcal{F}(X) \rightarrow \mathcal{F}(X/Z)$

The map  $p_*: \mathcal{F}(Y, T) \rightarrow \mathcal{F}(X, Z)$  is a weak equivalence in  $\mathcal{Y}_*$ .

### Basic examples of quadratic refinements of invariants

If  $X/k$  proper smooth, then  $X = \Sigma^+ X$  is strongly dualisable in the symmetric monoidal category  $\text{SH}(k)$ , we write its dual by  $X^\vee$ .

$$\text{Then } \chi(X/k) = (\mathbb{I}_k \xrightarrow{\text{unit}} \text{Hom}(X, X) \cong X \otimes X^\vee \xrightarrow[\text{evaluation}]{\text{counit}} \mathbb{I}_k)$$

$$\text{in } \text{End}_{\text{SH}(k)}(\mathbb{I}_k) = \text{GW}(k).$$

Which is called the cat. Euler-Char.

Serre's remark: this symmetric bilinear form is given by the Poincaré duality.

$$\chi(\mathbb{P}^n/k) = \begin{cases} \frac{n+1}{2} \mathbb{H} & n \text{ odd} \\ \langle 1 \rangle + \sum \mathbb{H} & n \text{ even} \end{cases}$$

$$\left( \begin{array}{l} \text{For } k = \mathbb{R} \subseteq \mathbb{C}, \\ \chi(X/\mathbb{R}) \xrightarrow[\text{sig}]{\text{EG}(\mathbb{R})} \chi(X(\mathbb{R})) \\ \chi(X(\mathbb{C})) \xrightarrow{\text{rank}} \chi(X(\mathbb{R})) \end{array} \right) = \mathbb{Z} \oplus \mathbb{Z}$$

# Theorem 1 (Jih-Y) Quadratic refinement of GOS formula

$\text{Char}(k) \neq 2$  and  $\dim X$  odd

$$Z \hookrightarrow X$$

$\downarrow p: \text{proper smooth.}$

$\text{Spec } k$

$K \in \text{SH}_c(X)$  such that

$K|_{X/Z}$  is dualizable, then

Quadratic artin conductor

$\Downarrow$

$$\chi(X, K) = \text{rk } K \cdot \chi(X/k) - \text{Art}(K) \quad \text{in } \mathcal{GW}(k)$$

$\parallel$

$$\chi(\mathbb{R}_x k)$$

$$\text{rk}(\text{ét realization of } K)$$

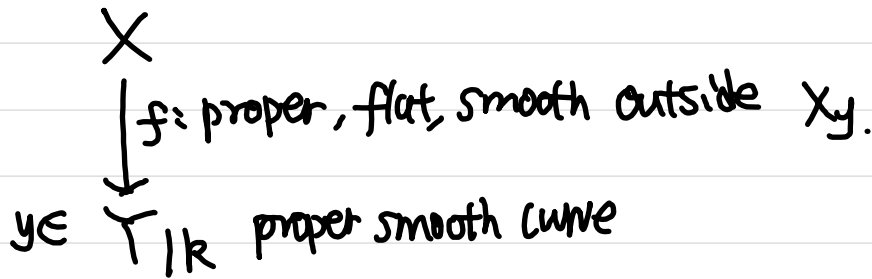
when  $X$  is a curve,  $Z$  = finite set of closed points,

taking rank + étale realization, get

$$\text{rk}(\text{Art}(K)) = \sum_{x \in Z} \alpha_x(\mathbb{R}_x k) \quad \text{Artin conductor of } \mathbb{R}_x k \text{ at } x \in Z$$

$$\alpha_x(\mathbb{F}) = \text{rank } \mathcal{F}_{\eta_x} - \text{rank } \mathcal{F}_{\bar{x}} + \text{Sw}_x \mathbb{F}$$

Theorem 2 (Jin-Y) Quadratic Bloch conductor formula.



Then we have

$$\begin{array}{c}
 \text{--- Art}(Rf_*\hat{F}) = f_! \left( \begin{array}{c} \text{a class in} \\ H_{MW} \Lambda(X_y/y) \end{array} \right) \text{ in } GW(k(y)) \\
 \uparrow \qquad \qquad \qquad \swarrow \quad \uparrow \qquad \qquad \qquad \downarrow \\
 \text{quadratic Artin} \qquad \qquad GW(k) \quad \text{Milnor-Witt spectrum} \qquad \qquad GW(k)
 \end{array}$$

Which generalize Bloch's conductor formula (conj 1987)

$$-\alpha_y(Rf_*\Lambda) = (-1)^{d+1} \deg(C_{d+1, X_y}^X (\Omega_{X/Y}^1) \cap [X])$$

$d = \text{rel. dim of } X \rightarrow Y.$

proved under

- $\left\{ \begin{array}{l} d=1 \text{ Bloch} \\ \text{higher dim 2020 by T. Saito} \\ (X_y)_{\text{red}} \text{ NCD, Kato-Saito} \\ \text{reproved by Y-Zhao, 2022.} \end{array} \right.$

## § 2 Non-acyclicity Class

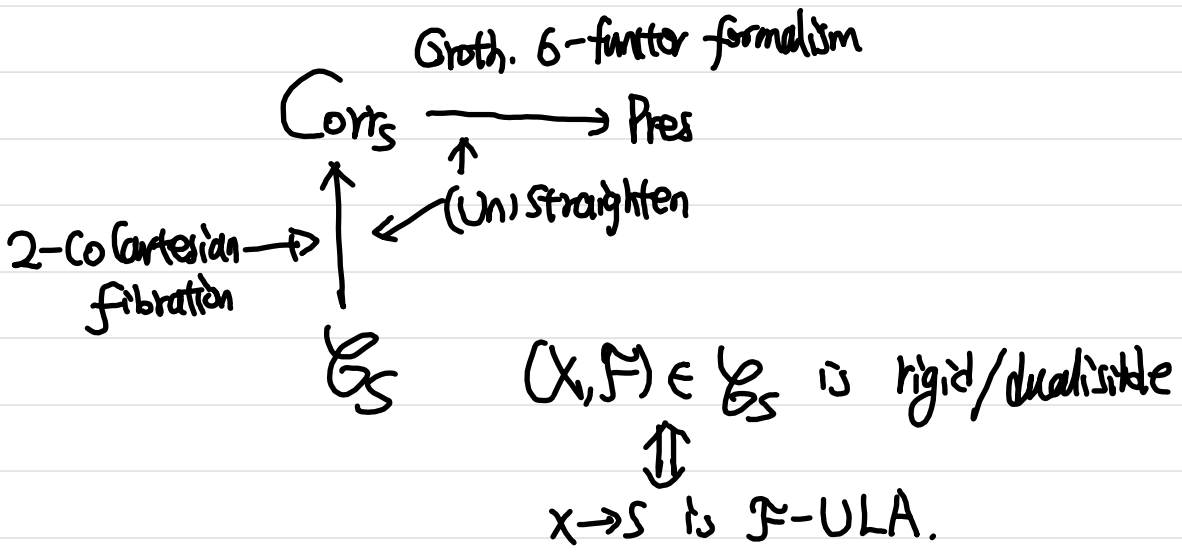
Notation  $\Delta = \left( \begin{array}{ccc} Z \hookrightarrow X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g \\ & S & \end{array} \right)$

$S$ : Noetherian scheme  
 $g$ : smooth

in  $Sch/S = \{ \text{sep of fit over } S \}$

Call  $Z = NA$  locus of  $\Delta$

$$D_c^b(\Delta) = \left\{ \mathcal{F} \in D_c^b(X, \mathcal{N}) \mid \begin{array}{l} h \text{ is } \mathcal{F}\text{-ULA} \\ f \text{ is } \mathcal{F}\text{-ULA outside } Z \end{array} \right\}$$



For any  $\mathcal{F} \in D_c^b(\Delta)$ , we will define

$$\mathcal{K}_\Delta \in D_c^b(X, \mathcal{N})$$

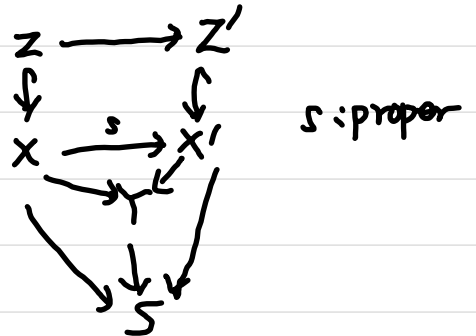
a class  $C_\Delta(\mathcal{F}) \in H_Z^0(X, \mathcal{K}_\Delta) \left( \cong H^0(Z, \mathcal{K}_{Z/S}) \right)$   
 when  $Z$  is small enough,  
 i.e.,  $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$ .

Thm (Y-Zhao, 2022)

(1) Given  $S' \xrightarrow{b} S$ , get  $\Delta'$  by base change, then

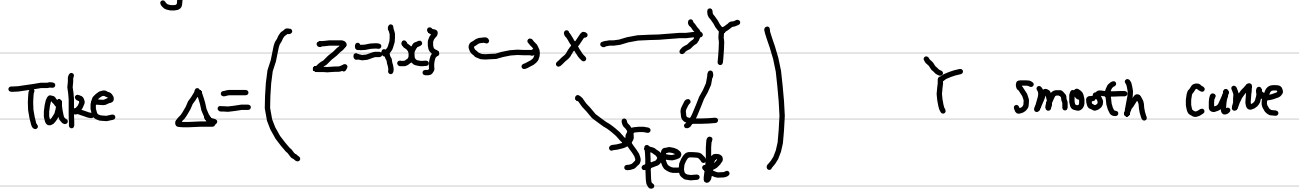
$$b_x^* C_\Delta(\mathcal{F}) = C_{\Delta'}(\mathcal{F}')$$

(2) Given  $\Delta \xrightarrow{\text{proper}} \Delta'$ , in the form



then  $s_* C_\Delta(\mathcal{F}) = C_{\Delta'}(s_* \mathcal{F})$

(3) cohomological Milnor formula

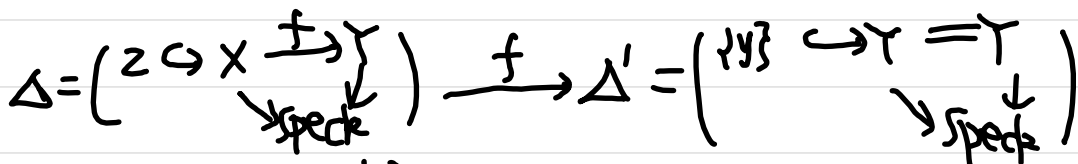


Then  $C_\Delta(\mathcal{F}) = -\dim \text{tot } R\mathbb{P}_{\tilde{x}}(\mathcal{F}, f)$ ,  $\mathcal{F} \in D_c^b(\Delta)$ .

$$H^0(\{x\}, \Lambda) = \Lambda \cdot \{x\}$$

cohomological conductor formula

(4) If  $X \xrightarrow{f} Y$  proper,  $Y$ : curve,  $y \in |Y|$ ,  $Z \subseteq f^{-1}(y)$



Apply (2) to  $\begin{array}{ccc} z & \longrightarrow & \{y\} \\ \downarrow & & \downarrow \\ x & \longrightarrow & Y \\ \searrow & & \downarrow \\ Y & = & \text{Spec } k \end{array}$  get  $f_* C_\Delta(\mathcal{F}) = C_{\Delta'}(f_* \mathcal{F}) \stackrel{\text{Milnor}}{=} -\dim \text{tot } R\mathbb{P}_y(\mathcal{F}, \text{id}) = -a_y(R f_* \mathcal{F})$

(5) Cohomological GOS formula (char  $k = p > 0$ )

$$\Delta = \left( \begin{array}{ccc} Z \hookrightarrow X & \xrightarrow{=} & X \\ & \searrow \text{Spec } k & \downarrow \end{array} \right), \quad X: \text{smooth curve}$$

$$\mathcal{F} \in D_c^b(\Delta) \iff \mathcal{F}|_{X \setminus Z} \text{ smooth}$$

$$C_\Delta(\mathcal{F}) = - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \text{ in } H^0(Z, \mathcal{K}_Z/k)$$

### Construction of $C_\Delta(\mathcal{F})$

$$\begin{array}{ccc} X' = X & \xrightarrow{i} & X \\ \downarrow \delta_1 & & \downarrow \delta_0 \\ X \times_Y X & \xrightarrow{i} & X \times_S X \\ \downarrow p & & \downarrow f \times f \\ Y & \xrightarrow{g} & Y \times_S Y \end{array} \quad f$$

For  $\mathcal{F} \in D_c^b(\Delta)$ , put

$$C_{Y/S}(\mathcal{F}) = \left( \delta_{0!} \Lambda = \delta_{0*} \Lambda \rightarrow \mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F}) \right) \\ \downarrow \\ \delta_{0*} \mathcal{K}_{Y/S}$$

We define a functor  $\delta^\Delta: D_c^b(X) \rightarrow D_c^b(X')$

$$i^* \mathcal{K} \otimes f^* \delta^! \Lambda \rightarrow i^! \mathcal{K} \rightarrow \delta^\Delta \mathcal{K} \xrightarrow{+1}$$

Technical Lemma  $\delta^\Delta C_{Y/S}(\mathcal{F})$  is supported on  $Z$

Then we define  $C_\Delta(\mathcal{F}) = \delta^\Delta C_{Y/S}(\mathcal{F})$ .

The most difficult part is to prove the following conjecture, formulated by myself with Zhao.

Conjecture If  $H^0(Z, \mathcal{K}_{Z/Y}) = H^1(Z, \mathcal{K}_{Z/Y}) = 0$ , then

$$C_{X/S}(\mathcal{F}) = C_r(f^* \Omega_{Y/S}^1) \cap C_{Y/S}(\mathcal{F}) + C_\Delta(\mathcal{F}) \text{ in } H^*(X, \mathcal{K}_{X/S}).$$

Y-zhao: true if  $Z = \emptyset$

Abbes-Saito: If  $f = \text{id}$ , and  $S = \text{Spec } k$ , OK.

Y-zhao:  $S = \text{Spec } k$ ,  $Y$ : smooth curve,  $Z$ : finite set of closed points.

$$C_{X/S}(\mathcal{F}) = C_1(f^* \Omega_{Y/k}^1) \cap C_{Y/S}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \left[ \begin{array}{l} \text{fibration} \\ \text{formula} \end{array} \right]$$

$$C_\Delta(\mathcal{F}) = - \sum_{x \in Z} \dim \text{Tot } R\Phi(\mathcal{F}, f)_* [x].$$

More conjecture (mixed version)

$$\blacksquare = \begin{array}{ccc} Z \hookrightarrow X & & \\ \downarrow \delta & \downarrow f & \\ T \hookrightarrow S & & \end{array} \quad D_C^b(\blacksquare) = \left\{ \mathcal{F} \in D_C^b(X, N) \mid \begin{array}{l} X \setminus Z \rightarrow S \\ \text{is } \mathcal{F}\text{-ULA} \end{array} \right\}$$

$\delta$  is a regular embedding. [是否可去掉  $T$ ]

Then  $\exists \mathcal{K}_{\blacksquare}$  and  $\exists$  class  $C_{\blacksquare}(\mathcal{F}) \in H_Z^*(X, \mathcal{K}_{\blacksquare})$  with similar properties.