SYSTEMS OF POINTS WITH COULOMB INTERACTIONS

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Abstract. Large ensembles of points with Coulomb interactions arise in various settings of condensed matter physics, classical and quantum mechanics, statistical mechanics, random matrices and even approximation theory, and give rise to a variety of questions pertaining to calculus of variations, Partial Differential Equations and probability. We will review these as well as "the mean-field limit" results that allow to derive effective models and equations describing the system at the macroscopic scale. We then explain how to analyze the next order beyond the mean-field limit, giving information on the system at the microscopic level. In the setting of statistical mechanics, this allows for instance to observe the effect of the temperature and to connect with crystallization questions.

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1. General setups

We are interested in large systems of points with Coulomb-type interactions, described through an energy of the form

$$\mathcal{H}_N(x_1,\dots,x_N) = \frac{1}{2}\sum_{i\neq j} \mathsf{g}(x_i-x_j) + N\sum_{i=1}^N V(x_i). \tag{1.1}$$

Here the points x_i belong to the Euclidean space $\mathbb{R}^{\mathsf{d}}$, although it is also interesting to consider points on manifolds. The interaction kernel $\mathsf{g}(x)$ is taken to be

$$(\mathsf{Log2}\text{ case})\quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=2, \tag{1.2}$$

$$(\mathsf{Coul}\text{ case})\quad \mathsf{g}(x) = \frac{1}{|x|^{\mathsf{d}-2}}, \quad \text{in dimension } \mathsf{d}\geq 3. \tag{1.3}$$

This is (up to a multiplicative constant) the Coulomb kernel in dimension $\mathsf{d}\geq 2$, i.e. the fundamental solution to the Laplace operator, solving

$$-\Delta \mathsf{g} = \mathsf{c}_{\mathsf{d}}\delta_0 \tag{1.4}$$

where δ_0 is the Dirac mass at the origin and c_{d} is an explicit constant depending only on the dimension. It is also interesting to broaden the study to the one-dimensional logarithmic case

$$(\mathsf{Log1}\text{ case})\quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=1, \tag{1.5}$$

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which is not Coulombian, and to more general Riesz interaction kernels of the form

$$\mathsf{g}(x) = \frac{1}{|x|^{\mathsf{s}}} \quad \mathsf{s} > 0. \tag{1.6}$$

The one-dimensional Coulomb interaction with kernel $-|x|$ is also of interest, but we will not consider it as it has been extensively studied and understood, see [113, 123, 124].

Finally, we have included a possible external field or confining potential V, which is assumed to be regular enough and tending to ∞ fast enough at ∞. The factor N in front of V makes the total confinement energy of the same order as the total repulsion energy, effectively balancing them and confining the system to a subset of $\mathbb{R}^{\mathsf{d}}$ of fixed size. Other choices of scaling would lead to systems of very large or very small size as $N \to \infty$.

The Coulomb interaction and the Laplace operator are obviously extremely important and ubiquitous in physics as the fundamental interactions of nature (gravitational and electromagnetic) are Coulombic. Coulomb was a French engineer and physicist working in the late 18th century, who did a lot of work on applied mechanics (such as modeling friction and torsion) and is most famous for his theory of electrostatics and magnetism. He is the first one who postulated that the force exerted by charged particles is proportional to the inverse distance squared, which corresponds in dimension $\mathsf{d} = 3$ to the gradient of the Coulomb potential energy $\mathsf{g}(x)$ as above. More precisely he wrote in [39] " It follows therefore from these three tests, that the repulsive force that the two balls [which were] electrified with the same kind of electricity exert on each other, follows the inverse proportion of the square of the distance." He developed a method based on systematic use of mathematical calculus (with the help of suitable approximations) and mathematical modeling (in contemporary terms) to predict physical behavior, systematically comparing the results with the measurements of the experiments he was designing and conducting himself. As such, he is considered as a pioneer of the "mathematization" of physics and in trusting fully the capacities of mathematics to transcribe physical phenomena [28].

Here we are more specifically focusing on Coulomb interactions between points, or in physics terms, discrete point charges. There are several mathematical problems that are interesting to study, all in the asymptotics of $N \to \infty$:

(1) understand minimizers and possibly critical points of (1.1) ;

(2) understand the statistical mechanics of systems with energy $\mathcal{H}_N$ and inverse temperature $\beta > 0$, governed by the so-called Gibbs measure

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N. \tag{1.7}$$

Here $\mathbb{P}_{N,\beta}$ is the density of probability of observing the system in the configuration $(x_1, \dots, x_N)$ if the inverse of the temperature is β. The constant $Z_{N,\beta}$ is called the "partition function" in physics, it is the normalization constant that makes $\mathbb{P}_{N,\beta}$ a probability measure, [1] i.e.

$$Z_{N,\beta} = \int_{(\mathbb{R}^{\mathsf{d}})^N} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N, \tag{1.8}$$

where the inverse temperature $\beta = \beta_N$ can be taken to depend on N, as there are several interesting scalings of β relative to N;

[1] One does not know how to explicitly compute the integrals (1.8) except in the particular case of (1.5) for specific V's where they are called Selberg integrals (cf. [72, 135])

(3) understand dynamic evolutions associated to (1.1), such as the gradient flow of HN given by the system of coupled ODEs

$$\dot{x_i} = -\frac{1}{N}\nabla_i \mathcal{H}_N(x_1, \dots, x_N), \tag{1.9}$$

the conservative dynamics given for instance in dimension 2 by the systems of ODEs

$$\dot{x_i} = \frac{1}{N}\nabla_i^\perp \mathcal{H}_N(x_1, \dots, x_N) \qquad \nabla^\perp = (-\partial_2, \partial_1), \tag{1.10}$$

or the Hamiltonian dynamics given by Newton's law

$$\ddot{x_i} = -\frac{1}{N}\nabla_i \mathcal{H}_N(x_1, \dots, x_N); \tag{1.11}$$

(4) understand the previous dynamic evolutions with temperature β^{-1} in the form of an added noise (Langevin-type equations) such as

$$dx_i = -\frac{1}{N}\nabla_i \mathcal{H}_N(x_1, \dots, x_N)dt + \sqrt{\beta^{-1}}dW_i \tag{1.12}$$

with W_i independent Brownian motions, or

$$dx_i = \frac{1}{N}\nabla_i^\perp \mathcal{H}_N(x_1, \dots, x_N)dt + \sqrt{\beta^{-1}}dW_i \tag{1.13}$$

in dimension 2 as above, or

$$dx_i = v_i dt \qquad dv_i = -\frac{1}{N}\nabla_i \mathcal{H}_N(x_1, \dots, x_N)dt + \sqrt{\beta^{-1}}dW_i. \tag{1.14}$$

From a mathematical point of view, the study of such systems touches on the fields of analysis (Partial Differential Equations and calculus of variations, approximation theory) particularly for **(1)**-**(3)**-**(4)**, probability (particularly for **(2)**-**(4)**), mathematical physics, and even geometry (when one considers such systems on manifolds or with curved geometries). Some of the crystallization questions they lead to also overlap with number theory as we will see below.

In the sequel we will mostly focus on the stationary settings **(1)** and **(2)**, while mentioning more briefly some known results about **(3)** and **(4)**, for which many questions remain open. Of course these various points are not unrelated, as for instance the Gibbs measure (1.7) can also be seen as an invariant measure for dynamics of the form (1.11) or (1.12).

The plan of the discussion is as follows: in the next section we review various motivations for studying such questions, whether from physics or within mathematics. In Section 3 we turn to the so-called "mean-field" or leading order description of systems **(1)** to **(4)** and review the standard questions and known results. We emphasize that this part can be extended to general interaction kernels $\mathbf{g}$, starting with regular (smooth) interactions which are in fact the easiest to treat. In Section 4, we discuss questions that can be asked and results that can be obtained at the next order level of expansion of the energy. This has only been tackled for problems **(1)** and **(2)**, and the specificity of the Coulomb interaction becomes important then.

2. Motivations

It is in fact impossible to list all possible topics in which such systems arise, as they are really numerous. We will attempt to give a short, necessarily biased, list of examples, with possible pointers to the relevant literature.

2.1. **Vortices in condensed matter physics and fluids.** In superconductors with applied magnetic fields, and in rotating superfluids and Bose-Einstein condensates, one observes the occurrence of quantized "vortices" (which are local point defects of superconductivity or superfluidity, surrounded by a current loop). The vortices repel each other, while being confined together by the effect of the magnetic field or rotation, and the result of the competition between these two effects is that, as predicted by Abrikosov [1], they arrange themselves in a particular *triangular lattice* pattern, called *Abrikosov lattice*, cf. Fig. 1 (for more pictures, see `www.fys.uio.no/super/vortex/`). Superconductors and superfluids are modelled by the

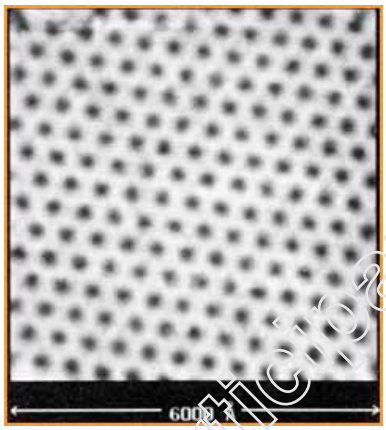

FIGURE 1. Abrikosov lattice, H. F. Hess et al. Bell Labs *Phys. Rev. Lett.* 62, 214 (1989)

celebrated Ginzburg-Landau energy [115], which in simplified form [2] can be written

$$\int |\nabla\psi|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2} \tag{2.1}$$

where ψ is a complex-valued unknown function (the "order parameter" in physics) and ε is a small parameter, and gives rise to the associated Ginzburg-Landau equation

$$\Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) = 0 \tag{2.2}$$

and its dynamical versions, the heat flow

$$\partial_t\psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) \tag{2.3}$$

and Schrödinger-type flow (also called the Gross-Pitaevskii equation)

$$i\partial_t\psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2). \tag{2.4}$$

When restricting to a two-dimensional situation, it can be shown rigorously (this was pioneered by [23] for (2.1) and extended to the full gauged model [25,158,159]) that the minimization of (2.1) can be reduced, in terms of the vortices and as $\varepsilon \to 0$, to the minimization of an energy of the form (1.1) in the case (1.2) (for a formal derivation, see also [169, Chap. 1]) and this naturally leads to the question of understanding the connection between minimizers

[2] The complete form for superconductivity contains a gauge-field, but we omit it here for simplicity.

of (1.1) + (1.2) and the Abrikosov triangular lattice. Similarly, the dynamics of vortices under (2.3) can be formally reduced to (1.9), respectively under (2.4) to (1.10). This was established formally for instance in [64, 142] and proven for a fixed number of vortices N and in the limit $\varepsilon \to 0$ in [24, 54, 55, 106, 127–129] until the first collision time and in [21, 22, 165, 176] including after collision.

Vortices also arise in classical fluids, where in contrast with what happens in superconductors and superfluids, their charge is not quantized. In that context the energy (1.1)+(1.2) is sometimes called the Kirchhoff energy and the system (1.10), known as the point-vortex system, corresponds to the dynamics of idealized vortices in an incompressible fluid whose statistical mechanics analysis was initiated by Onsager, cf. [67] (one of the motivations for studying (1.13) is precisely to understand fluid turbulence as he conceived). It has thus been quite studied as such, see [132] for further reference. The study of evolutions like (1.11) is also motivated by plasma physics in which the interaction between ions is Coulombic, cf. [100].

2.2. **Fekete points and approximation theory.** Fekete points arise in interpolation theory as the points minimizing interpolation errors for numerical integration [157]. More precisely, if one is looking for N interpolation points $\{x_1, \dots, x_N\}$ in K such that the relation

$$\int_K f(x)dx = \sum_{j=1}^N w_j f(x_j)$$

is exact for the polynomials of degree $\leq N-1$, one sees that one needs to compute the coefficients w_j such that $\int_K x^k = \sum_{j=1}^N w_j x_j^k$ for $0 \leq k \leq N-1$, and this computation is easy if one knows to invert the Vandermonde matrix of the $\{x_j\}_{j=1\dots N}$. The numerical stability of this operation is as large as the *condition number* of the matrix, i.e. as the Vandermonde determinant of the $(x_1, \dots, x_N)$. The points that minimize the maximal interpolation error for general functions are easily shown to be the Fekete points, defined as those that maximize

$$\prod_{i \neq j} |x_i - x_j|$$

or equivalently minimize

$$-\sum_{i \neq j} \log |x_i - x_j|.$$

They are often studied on manifolds, such as the d-dimensional sphere. In Euclidean space, one also considers "weighted Fekete points" which maximize

$$\prod_{i<j} |x_i - x_j| e^{-N \sum_i V(x_i)}$$

or equivalently minimize

$$-\frac{1}{2} \sum_{i \neq j} \log |x_i - x_j| + N \sum_{i=1}^N V(x_i)$$

which in dimension 2 corresponds exactly to the minimization of $\mathcal{H}_N$ in the particular case $\mathsf{Log2}$. They also happen to be zeroes of orthogonal polynomials, see [174].

Since $-\log|x|$ can be obtained as $\lim_{s\to 0} \frac{1}{s}(|x|^{-s} - 1)$, there is also interest in studying "Riesz s-energies", i.e. the minimization of

$$\sum_{i \neq j} \frac{1}{|x_i - x_j|^{\mathsf{s}}} \tag{2.5}$$

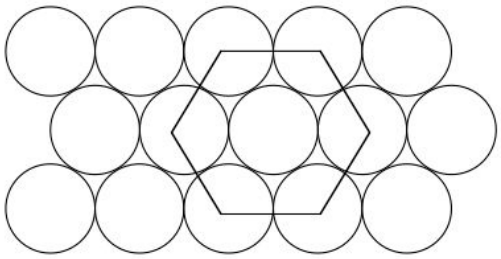

FIGURE 2. The triangular lattice solves the sphere packing problem in dimension 2

for all possible s, hence a motivation for (1.6). Varying s from 0 to ∞ connects Fekete points to the optimal sphere packing problem, which formally corresponds to the minimization of (2.5) with $\mathsf{s} = \infty$. The optimal packing problem is solved in 1, 2 and 3 dimensions. The solution in dimension 2 is the triangular lattice [68] (i.e. the same as the Abrikosov lattice), see Figure 2, in dimension 3 it is the FCC (face-centered cubic) lattice [91]. In higher dimension, the solution is in general not known except for dimensions 8 and 24 where it has been solved in a recent remarkable breakthrough [53, 187] (we refer the reader to the nice presentation in [51] and the review [175]). In high dimension, where the problem is important for error-correcting codes, it is expected that the solution is *not* a lattice.

For these aspects, we refer to the the review papers [37, 156] and references therein.

2.3. **Statistical mechanics and quantum mechanics.** The ensemble given by (1.7) in the Log2 case is called in physics a two-dimensional *Coulomb gas* or *one-component plasma* and is a classical ensemble of statistical mechanics (see e.g. [2, 103, 104, 108, 110, 162]). Two-dimensional Coulomb interactions also arise in quantum mechanics: the Coulomb gas is relevant to understand the fractional quantum Hall effect [155, 180] since it is in direct correspondance with Laughlin's wave function (this is the "plasma analogy"), cf. [85, 116, 117]. For recent mathematical progress using this correspondence, cf. [151, 153, 154]. The two-dimensional Coulomb gas with $\beta = 2$ also arises as the wave-function density of the ground state for the system of N non-interacting fermions confined to a plane with a perpendicular magnetic field [72, Chap. 15]. The Coulomb case with $\mathsf{d} = 3$ can be seen as a toy (classical) model for matter (see e.g. [104, 125, 126, 141]). The 1-dimensional log gas Log1 also arises as the wave-function density in several quantum mechanics systems: examples are the Tonks-Girardeau model of "impenetrable" Bosons [74, 84], the Calogero-Sutherland quantum many-body Hamiltonian [72, 73] and finally the density of the many-body wave function of non-interacting fermions in a harmonic trap [56]. It also arises in several non-intersecting paths models from probability, cf. [72].

The general Riesz case can be seen as a generalization of the Coulomb case, motivations for studying Riesz gases are numerous in the physics literature (in solid state physics, ferrofluids, elasticity), see for instance [10, 43, 133, 184], they can also correspond to systems with Coulomb interaction constrained to a lower-dimensional subspace.

In all cases of interactions, the systems governed by the Gibbs measure $\mathbb{P}_{N,\beta}$ are considered as difficult systems of statistical mechanics because the interactions are truly long-range, singular, and the points are not constrained to live on a lattice.

As always in statistical mechanics [98], one would like to understand if there are phase-transitions for particular values of the (inverse) temperature β in the large volume limit.

For the systems studied here, one may expect, after a suitable blow-up of the system, what physicists call a liquid for small β, and a crystal for large β. The meaning of crystal in this instance is not to be taken literally as a lattice, but rather as a system of points whose 2-point correlation function $\rho^{(2)}(x,y)$ does not decay too fast as $x-y\to\infty$. A phase-transition at finite β has been conjectured in the physics literature for the Log2 case (see e.g. [38, 42, 49]) but its precise nature is still unclear (see e.g. [179] for a discussion).

2.4. **Two component plasma case.** The two-dimensional "one component plasma", consisting of positively charged particles, has a "two-component" counterpart which consists in N particles $x_1,\dots,x_N$ of charge $+1$ and N particles $y_1,\dots,y_N$ of charge -1 interacting logarithmically, with energy

$$\mathcal{H}_N(x_1,\dots,x_N,y_1,\dots,y_N)=-\sum_{i\neq j}\log|x_i-x_j|-\sum_{i\neq j}\log|y_i-y_j|+\sum_{i,j}\log|x_i-y_j|$$

and the Gibbs measure

$$\frac{1}{Z_{N,\beta}}e^{-\beta\mathcal{H}_N(x_1,\dots,x_N,y_1,\dots,y_N)}dx_1\dots dx_N\,dy_1\dots dy_N.$$

Although the energy is unbounded below (positive and negative points attract), the Gibbs measure is well defined for β small enough, more precisely the partition function converges for $\beta<2$. The system is then seen to form dipoles of oppositely charged particles which attract but do not collapse, thanks to the thermal agitation. The two-component plasma is interesting due to its close relation to two important theoretical physics models: the XY model and the sine-Gordon model (cf. the review [177]), which exhibit a Kosterlitz-Thouless phase transition [26] consisting in the binding of these "vortex-antivortex" dipoles. For further reference, see [60, 76, 77, 90].

2.5. **Random matrix theory.** The study of (1.7) has attracted a lot of attention due to its connection with random matrix theory (we refer to [72] for a comprehensive treatment). Random matrix theory (RMT) is a relatively old theory, pionereed by statisticians and physicists such as Wishart, Wigner and Dyson, and originally motivated by the study of sample covariance matrices for the former and the understanding of the spectrum of heavy atoms for the two latter, see [135]. For more recent mathematical reference see [9, 57, 72]. The main question asked by RMT is : what is the law of the spectrum of a large random matrix ? As first noticed in the foundational papers of [63, 189], in the particular cases (1.5) and (1.2) the Gibbs measure (1.7) corresponds in some particular instances to the joint law of the eigenvalues (which can be computed algebraically) of some famous random matrix ensembles:

- for Log2, $\beta=2$ and $V(x)=|x|^2$, (1.7) is the law of the (complex) eigenvalues of an $N\times N$ matrix where the entries are chosen to be normal Gaussian i.i.d. This is called the Ginibre ensemble.
- for Log1, $\beta=2$ and $V(x)=x^2/2$, (1.7) is the law of the (real) eigenvalues of an $N\times N$ Hermitian matrix with complex normal Gaussian iid entries. This is called the Gaussian Unitary Ensemble.
- for Log1, $\beta=1$ and $V(x)=x^2/2$, (1.7) is the law of the (real) eigenvalues of an $N\times N$ real symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Orthogonal Ensemble.

- for Log1, $\beta = 4$ and $V(x) = x^2/2$, (1.7) is the law of the eigenvalues of an $N \times N$ quaternionic symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Symplectic Ensemble.
- the general-β case of Log1 can also be represented, in a slightly more complicated way, as a random matrix ensemble [62, 111].

One thus observes in these ensembles the phenomenon of "repulsion of eigenvalues": they repel each other logarithmically, i.e. like two-dimensional Coulomb particles.

The stochastic evolution (1.12) in the case Log1 is exactly the Dyson Brownian motion, which is of particular importance in random matrices since the GUE process is the invariant measure for this evolution, it has served to prove universality for the statistics of eigenvalues of general Wigner matrices, i.e. those with iid but not necessarily Gaussian entries, see [66] (and [182] for another approach), and has thus been studied with that perspective, see for instance [97] and references therein.

For the Log1 and Log2 cases, at the specific temperature $\beta = 2$, the law (1.7) acquires a special algebraic feature : it becomes a *determinantal* process, part of a wider class of processes (see [30, 96]) for which the correlation functions are explicitly given by certain determinants. This allows for many explicit algebraic computations, and is part of *integrable probability* on which there is a large literature [31].

2.6. **Complex geometry and theoretical physics.** Two-dimensional Coulomb systems (in the determinantal case $\beta = 2$) are of interest to geometers because they serve to construct Kähler-Einstein metrics with positive Ricci curvature on complex manifolds, cf. [17, 18]. Another important motivation is the construction of Laughlin states for the Fractional Quantum Hall effect on complex manifolds, which effectively reduces to the study of a two-dimensional Coulomb gas on manifolds. The coefficients in the expansion of the (logarithm of the) partition function have interpretations as geometric invariants, cf. for instance [112].

3. The mean field limits and macroscopic behavior

3.1. **Questions.** The first question that naturally arises is to understand the limits as $N \to \infty$ of the *empirical measure* defined by [3]

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \tag{3.1}$$

for configurations of points that minimize the energy (1.1), critical points, solutions of the evolution problems, or typical configurations under the Gibbs measure (1.7), thus hoping to derive effective equations or minimization problems that describe the average or mean-field behavior of the system. The term mean-field refers to the fact that, from the physics perspective, each particle feels the collective field generated by all the other particles, averaged by dividing it by the number of particles. That collective field is $\mathsf{g} * \mu_N$, except that it is singular at each particle, so to evaluate it at x_i one would first have to remove the contribution of x_i itself.

Another point of view is that of correlation functions. One may denote by

$$\rho_N^{(k)}(x_1, \dots, x_k) \tag{3.2}$$

[3] Note that the configurations contain N points which also implicitly depend on N themselves, but we do not keep track of this dependence for the sake of lightness of notation.

the k-point correlation function, which is the probability density (for each specific problem) of observing a particle at x_1, a particle at x_2, ..., and a particle at x_k (these functions should of course be symmetric with respect to permutation of the labels). For instance, in the case (1.7), $\rho_N^{(N)}$ is simply $\mathbb{P}_{N,\beta}$ itself, and the $\rho_N^{(k)}$ are its marginals (obtained by integrating $\mathbb{P}_{N,\beta}$ with respect to all its variables but k). One then wants to understand the limit as $N \to \infty$ of each $\rho_N^{(k)}$, with fixed k. Mean-field results will typically imply that the limiting $\rho^{(k)}$'s have a factorized form

$$\rho^{(k)}(x_1, \dots, x_k) = \mu(x_1) \dots \mu(x_k) \tag{3.3}$$

for the appropriate μ which is also equal to $\rho^{(1)}$. This is called *molecular chaos* according to the terminology introduced by Boltzmann, and can be interpreted as the particles becoming independent in the limit. When looking at the dynamic evolutions of problems **(3)** and **(4)**, starting from initial data for which $\rho^{(k)}(0, \cdot)$ are in such a factorized form, one asks whether this remains true for $\rho^{(k)}(t, \cdot)$ for $t > 0$, if so this is called *propagation of (molecular) chaos*. It turns out that the convergence of the empirical measure (3.1) to a limit μ and the fact that each $\rho^{(k)}$ can be put in factorized form are essentially equivalent, see [88, 95] and references therein — ideally, one would also like to find quantitative rates of convergences in N, and they will typically deteriorate as k gets large. In the following we will focus on the mean-field convergence approach, via the empirical measure.

In the statistical mechanics setting **(2)**, the quest for estimates on $Z_{N,\beta}$ as $N \to \infty$ is also a constant theme. Indeed, the quantity $-\beta^{-1} \log Z_{N,\beta}$ is called the *free energy*, and its dependence on β encodes a lot of the physical quantities of the system. For instance, points of non-differentiability of $\log Z_{N,\beta}(\beta)$ are interpreted as phase-transitions.

3.2. **The equilibrium measure.** The leading order behavior of $\mathcal{H}_N$ is related to the functional

$$\mathcal{I}_V(\mu) := \frac{1}{2} \iint_{\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^{\mathsf{d}}} V(x) d\mu(x) \tag{3.4}$$

defined over the space $\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ of probability measures on $\mathbb{R}^{\mathsf{d}}$ (which may also take the value $+\infty$). This is something one may naturally expect since $\mathcal{I}_V(\mu)$ appears as the continuum version of the discrete energy $\mathcal{H}_N$. From the point of view of statistical mechanics, $\mathcal{I}_V$ is the "mean-field" limit energy of $\mathcal{H}_N$, while from the point of view of probability, $\mathcal{I}_V$ plays the role of a *rate function*.

Assuming some lower semi-continuity of V and that it grows faster than g at ∞, it was shown in [78] that the minimum of $\mathcal{I}_V$ over $\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ exists, is finite and is achieved by a unique μ_V (unique by strict convexity of $\mathcal{I}_V$), which has compact support and a density, and is uniquely characterized by the fact that there exists a constant c such that

$$\begin{cases} h^{\mu_V} + V \geq c & \text{in } \mathbb{R}^{\mathsf{d}} \\ h^{\mu_V} + V = c & \text{in the support of } \mu_V \end{cases} \tag{3.5}$$

where

$$h^{\mu_V}(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y) d\mu_V(y) \tag{3.6}$$

is the "electrostatic" potential generated by μ_V.

This measure μ_V is called the (Frostman) *equilibrium measure*, and the result is true for more general repulsive kernels than Coulomb, for instance for all regular kernels or inverse powers of the distance which are integrable.

Example 3.1. When g is the Coulomb kernel, applying the Laplacian on both sides of (3.5) gives that, in the interior of the support of the equilibrium measure, if $V \in C^2$,

$$\mathsf{c}_\mathsf{d}\mu_V = \Delta V \tag{3.7}$$

i.e. the density of the measure on the interior of its support is given by $\frac{\Delta V}{\mathsf{c}_\mathsf{d}}$. For example if V is quadratic, this density is constant on the interior of its support. If $V(x) = |x|^2$ then by symmetry μ_V is the indicator function of a ball (up to a multiplicative factor), this is known as the *circle law* for the Ginibre ensemble in the context of Random Matrix Theory. An illustration of the convergence to this circle law can be found in Figure 3. In dimension $\mathsf{d} = 1$, with $\mathsf{g} = -\log|\cdot|$ and $V(x) = x^2$, the equilibrium measure is $\mu_V(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}$, which corresponds in the context of RMT (GUE and GOE ensembles) to the famous *Wigner semi-circle law*, cf. [135, 189].

In the Coulomb case, the equilibrium measure μ_V can also be interpreted in terms of the solution to a classical *obstacle problem* (and in the Riesz case (1.6) with $\mathsf{d}-2 \leq \mathsf{s} < \mathsf{d}$ a "fractional obstacle problem"), which is essentially dual to the minimization of $\mathcal{I}_V$, and better studied from the PDE point of view (in particular the regularity of μ_V and of the boundary of its support). For this aspect, see [169, Chap. 2] and references therein.

Frostman's theorem is the basic result of potential theory. The relations (3.5) can be seen as the Euler-Lagrange equations associated to the minimization of $\mathcal{I}_V$. They state that in the static situation, the total potential, sum of the potential generated by μ_V and the external potential V must be constant in the support of μ_V, i.e. in the set where the "charges" are present.

More generally $\nabla(h^\mu + V)$ can be seen as the total "mean-field force" acting on charges with density μ (i.e. each particle feels the average collective force generated by the other particles), and for the particle to be at rest one needs that force to vanish. Thus $\nabla(h^\mu + V)$ should vanish on the support of μ, in fact the stationarity condition that formally emerges as the limit for critical points of $\mathcal{H}_N$ is

$$\nabla(h^\mu + V)\mu = 0. \tag{3.8}$$

The problem with this relation is that the product $\nabla h^\mu \mu$ does not always make sense, since a priori μ is only a probability measure and h^μ is not necessarily continuous, however, in dimension 2, one can give a weak form of the equation which always makes sense, inspired by Delort's work in fluid mechanics [58], cf. [158, Chap. 13].

3.3. Convergence of minimizers.

Theorem 1. *We have*

$$\lim_{N\to\infty} \frac{\min \mathcal{H}_N}{N^2} = \min \mathcal{I}_V = \mathcal{I}_V(\mu_V) \tag{3.9}$$

and if $(x_1, \dots, x_N)$ *minimize* $\mathcal{H}_N$ *then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \rightharpoonup \mu_V \tag{3.10}$$

in the weak sense of probability measures.

This result is usually attributed to [50], one may see the proof in [157] for the logarithmic cases, the general case can be treated exactly in the same way [169, Chap. 2], and is valid for very general interactions g (for instance radial decreasing and integrable near 0). In modern language it can be phrased as a Γ-convergence result. It can also easily be expressed in terms of convergence of marginals, as a molecular chaos result.

3.4. **Parallel results for Ginzburg-Landau vortices.** The analogue mean field result and leading order asymptotic expansion of the minimal energy has also been obtained for the two-dimensional Ginzburg-Landau functional of superconductivity (2.1), see [158, Chap. 7]. It is phrased as the convergence of the vorticity $\nabla \times \langle i\psi, \nabla\psi\rangle$, normalized by the proper number of vortices, to an equilibrium measure, or the solution to an obstacle problem. The analogue of (3.8) is also derived for critical points in [158, Chap. 13], where an appropriate weak sense for this relation is given.

3.5. **Deterministic dynamics results - problems (3).** For general reference on problems of the form **(3)** and **(4)**, we refer to [178]. In view of the above discussion, in the dynamical cases (1.9) or (1.10), one expects as analogue results the convergences of the empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ to probability densities μ that satisfy the limiting mean-field evolutions

$$\partial_t \mu = -\mathrm{div}\,(\nabla(h^\mu + V)\mu) \tag{3.11}$$

respectively

$$\partial_t \mu = -\mathrm{div}\,(\nabla^\perp(h^\mu + V)\mu) \tag{3.12}$$

where again $h^\mu = \mathsf{g} * \mu$ as in (3.6). These are nonlocal transport equations where the density μ is transported along the velocity field $-\nabla(h^\mu + V)$, i.e. advected by the mean-field force that the distribution generates.

In the two-dimensional Coulomb case (1.2) with $V = 0$, (3.12) is also well-known as the vorticity form of the incompressible Euler equation, describing the evolution of the vorticity in an ideal fluid, with velocity given by the Biot-Savart law. As such, this equation is well-studied in this context, and the convergence of solutions of (1.10) to (3.12), also known as the point-vortex approximation to Euler, has been rigorously proven, see [89, 164].

As for (3.11), it is a dissipative equation, that can be seen as a gradient flow on the space of probability measures equipped with the so-called Wasserstein W_2 (or Monge-Kantorovitch) metric. In the dimension 2 logarithmic case, it was first introduced by Chapman-Rubinstein-Schatzman [47] and E [65] as a formal model for superconductivity, and in that setting the gradient flow description has been made rigorous (see [6]) using the theory of gradient flows in metric spaces of [5, 140]. The equation can also be studied by PDE methods [130, 172]. The derivation of this gradient flow equation (3.11) from (2.3) can be guessed by variational arguments, i.e. "Γ-convergence of gradient flows", see [166]. The analogue of the rigorous passage from (1.9) or (1.10) to (3.11) or (3.12) has been accomplished at the level of the full parabolic and Schrödinger Ginzburg-Landau PDEs (2.3) and (2.4) [105, 114, 170]. The proof in the third paper relies on a "modulated energy" argument which consists in finding a suitable energy, modelled on the Ginzburg-Landau energy, which measures the distance to the desired limiting solution, and for which a Gronwall inequality can be shown to hold.

Convergence of solutions to (1.9)-(1.10) to solutions of (3.11), resp. (3.12), in general dimensions is in general not known, this is due to the singularity of the Coulomb interaction combined with the nonlinear character of the product $h^\mu\mu$ (and its discrete analogue) which prevents from directly taking limits in the equation. Positive results on mean field limits and

propagation of chaos exist for less singular interactions [93] or in dimension 1 [19]. Progress has also been made for a class of Riesz interactions (more singular than Coulomb) in dimensions 1 and 2 in [61] by directly adapting the modulated energy approach of [170] to the discrete setting.

As far as (1.11) is concerned, the limiting equation is formally found to be the Vlasov-Poisson equation

$$\partial_t \rho + v \cdot \nabla_x \rho + \nabla(h^\mu + V) \cdot \nabla_v \rho = 0 \tag{3.13}$$

where $\rho(t,x,v)$ is the density of particles at time t with position x and velocity v, and $\mu(t,x) = \int \rho(t,x,v)dv$ is the density of particles. The rigorous convergence of (1.11) to (3.13) and propagation of chaos are not proven in all generality (i.e. for all initial data) but it has been established in a statistical sense (i.e. randomizing the initial condition) and often truncating the interactions, see [29,94,109,118,119] and also the reviews on the topic [88,100].

Overall, much remains open in this class of problems, even at the mean field level and how to treat singular interactions such as the Coulomb one is only known in the conservative cases.

3.6. **Noisy dynamics - problems (4).** The noise terms in these equations gives rise to an additive Laplacian term in the limiting equations. For instance the limiting equation for (1.12) is expected to be the McKean equation

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \mathrm{div}\,(\nabla(h^\mu + V)\mu) \tag{3.14}$$

and the convergence is known for regular interactions since the seminal work of [134], see also the reviews [100, 181].

For singular interactions, the situation has been understood for the one-dimensional logarithmic case [44], then for all Riesz interactions (1.6) [19]. Higher dimensions with singular interactions is largely open. It is expected that the noise should help the convergence and propagation of chaos, but an appropriate method still remains elusive.

For the conservative case (1.13) the limiting equation is a viscous conservative equation of the form

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \mathrm{div}\,(\nabla^\perp(h^\mu + V)\mu) \tag{3.15}$$

which in the two-dimensional logarithmic case (1.2) is the Navier-Stokes equation in vorticity form. The convergence in that particular case was established in [75]. Recent progress of [101] allows to treat quite rough interactions (including Coulomb) and prove convergence in an appropriate statistical sense.

For the case of (1.14), the limiting equation is the McKean-Vlasov equation

$$\partial_t \rho + v \cdot \nabla_x \rho + \nabla(h^\mu + V) \cdot \nabla_v \rho - \frac{1}{\beta}\Delta\rho = 0 \tag{3.16}$$

with the same notation as for (3.13), and convergence in the case of bounded-gradient kernels is proven in [102], see also references therein.

3.7. **With temperature: statistical mechanics.** Let us now turn to problem **(2)** and consider the situation with temperature as described via the Gibbs measure (1.7). One can determine that two temperature scaling choices are interesting: the first is taking β independent of N, the second is taking $\beta_N = \frac{\beta}{N}$ with some fixed β. In the former, which can

be considered a "low temperature" regime, the behavior of the system is still governed by the equilibrium measure μ_V. The result can be phrased using the language of Large Deviations Principles (LDP), cf. [59] for definitions and reference.

Theorem 2. *The sequence $\{\mathbb{P}_{N,\beta}\}_N$ of probability measures on $\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ satisfies a large deviations principle at speed N^2 with good rate function $\beta\hat{\mathcal{I}}_V$ where $\hat{\mathcal{I}}_V = \mathcal{I}_V - \min_{\mathcal{P}(\mathbb{R}^{\mathsf{d}})} \mathcal{I}_V = \mathcal{I}_V - \mathcal{I}_V(\mu_V)$. Moreover*

$$\lim_{N\to+\infty} \frac{1}{N^2} \log Z_{N,\beta} = -\beta \mathcal{I}_V(\mu_V) = -\beta \min_{\mathcal{P}(\mathbb{R}^{\mathsf{d}})} \mathcal{I}_V. \tag{3.17}$$

The concrete meaning of the LDP is that if E is a subset of the space of probability measures $\mathcal{P}(\mathbb{R}^{\mathsf{d}})$, after identifying configurations $(x_1, \dots, x_N)$ in $(\mathbb{R}^{\mathsf{d}})^N$ with their empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, we may write

$$\mathbb{P}_{N,\beta}(E) \approx e^{-\beta N^2(\min_E \mathcal{I}_V - \min \mathcal{I}_V)}, \tag{3.18}$$

which in view of the uniqueness of the minimizer of $\mathcal{I}_V$ implies that configurations whose empirical measure does not converge to μ_V as $N \to \infty$ have exponentially decaying probability. In other words the Gibbs measure concentrates as $N \to \infty$ on configurations for which the empirical measure is very close to μ_V, i.e. the temperature has no effect on the mean-field behavior.

This result was proven in the logarithmic cases in [145] (in dimension 2), [15] (in dimension 1) and [16] (in dimension 2) for the particular case of a quadratic potential (and $\beta = 2$), see also [18] for results in a more general (still determinantal) setting of multidimensional complex manifolds, or [45] which recently treated more general singular g's and V's. It is actually valid in any dimension, and is not at all specific to the Coulomb interaction (the proof works as well for more general interaction potentials, see [169]).

In the high-temperature regime $\beta_N = \frac{\beta}{N}$, the temperature is felt at leading order and brings an entropy term. More precisely there is a temperature-dependent equilibrium measure $\mu_{V,\beta}$ which is the unique minimizer of

$$I_{V,\beta}(\mu) = \beta \mathcal{I}_V(\mu) + \int \mu \log \mu. \tag{3.19}$$

Contrarily to the equilibrium measure, $\mu_{V,\beta}$ is not compactly supported, but decays exponentially fast at infinity. This mean-field behavior and convergence of marginals was first established for logarithmic interactions [41, 108] (see [136] for the case of regular interactions) using an approach based on de Finetti's theorem. In the language of Large Deviations, the same LDP as above then holds with rate function $I_{V,\beta} - \min I_{V,\beta}$, and the Gibbs measure now concentrates as $N \to \infty$ on a neighborhood of $\mu_{V,\beta}$, for a proof see [79]. Again the Coulomb nature of the interaction is not really needed. One can also refer to [149, 150] for the mean-field and chaos aspects with a particular focus on their adaptation to the quantum setting.

4. Beyond the mean field limit : next order study

We have seen that studying systems with Coulomb (or more general) interactions at leading order leads to a good understanding of their limiting macroscopic behavior. One would like to go further and describe their microscopic behavior, at the scale of the typical interdistance between the points, $N^{-1/\mathsf{d}}$. This in fact comes as a by-product of a next-to-leading

order description of the energy $\mathcal{H}_N$, which also comes together with a next-to-leading order expansion of the free energy in the case (1.7).

Thinking of energy minimizers or of typical configurations under (1.7), since one already knows that $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ is small, one knows that the so-called *discrepancy* in balls $B_r(x)$ for instance, defined as

$$D(x,r) := \int_{B_r(x)} \sum_{i=1}^N \delta_{x_i} - N\, d\mu_V$$

is $o(r^{\mathsf{d}}N)$ as long as $r > 0$ is fixed. Is this still true at the mesoscopic scales for r of the order $N^{-\alpha}$ with $\alpha < 1/\mathsf{d}$? Is it true down to the microscopic scale, i.e. for $r = RN^{-1/\mathsf{d}}$ with $R \gg 1$? Does it hold regardless of the temperature? This would correspond to a *rigidity result*. Note that point processes with discrepancies growing like the perimeter of the ball have been called *hyperuniform* and are of interest to physicists for a variety of applications, cf. [184], see also [82] for a review of the link between rigidity and hyperuniformity. An addition question is: how much of the microscopic behavior depends on V or in another words is there a form of universality in this behavior? Such questions had only been answered in details in the one-dimensional case (1.5) as we will see below.

4.1. **Expanding the energy to next order.** The first step that we will describe is how to expand the energy $\mathcal{H}_N$ around the measure $N\mu_V$, following the approach initiated in [161] and continued in [121, 144, 152, 160]. It relies on a splitting of the energy into a fixed leading order term and a next order term expressed in terms of the charge fluctuations, and on a rewriting of this next order term via the "electric potential" generated by the points. More precisely, exploiting the quadratic nature of the interaction, and letting $\triangle$ denote the diagonal in $\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}$, let us expand

$$\begin{aligned}
\mathcal{H}_N(x_1,\dots,x_N) &= \frac12 \sum_{i\neq j} \mathsf{g}(x_i-x_j) + N\sum_{i=1}^N V(x_i)\\
&= \frac12 \iint_{\triangle^c} \mathsf{g}(x-y) d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x) d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x)\\
&= \frac{N^2}{2} \iint_{\triangle^c} \mathsf{g}(x-y) d\mu_V(x) d\mu_V(y) + N^2 \int_{\mathbb{R}^{\mathsf{d}}} V d\mu_V\\
&+ N\iint_{\triangle^c} \mathsf{g}(x-y) d\mu_V(x) \Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)\\
&+ \frac12 \iint_{\triangle^c} \mathsf{g}(x-y) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y). \qquad (4.1)
\end{aligned}$$

Recalling that μ_V is characterized by (3.5), we see that the middle term

$$\begin{aligned}
(4.2)\quad N\iint_{\triangle^c} \mathsf{g}(x-y) d\mu_V(x) d(\sum_{i=1}^N \delta_{x_i} - N\mu_V)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d(\sum_{i=1}^N \delta_{x_i} - N\mu_V)\\
= N\int_{\mathbb{R}^{\mathsf{d}}} (h^{\mu_V} + V) d(\sum_{i=1}^N \delta_{x_i} - N\mu_V)
\end{aligned}$$

can be considered as vanishing (at least it does if all the points x_i fall in the support of μ_V). We are then left with

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}_V(\mu_V) + F_N^{\mu_V}(x_1,\dots,x_N) \tag{4.3}$$

with

$$F_N^{\mu_V}(x_1,\dots,x_N) = \frac{1}{2}\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y). \tag{4.4}$$

The relation (4.3) is a next-order expansion of $\mathcal{H}_N$ (cf. (3.9)), valid for arbitrary configurations. The "next-order energy" $F_N^{\mu_V}$ can be seen as the Coulomb energy of the neutral system formed by the N positive point charges at the x_i's and the diffuse negative charge $-N\mu_V$ of same mass. To further understand $F_N^{\mu_V}$ let us introduce the potential generated by this system, i.e.

$$H_N(x) = \int_{\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) \tag{4.5}$$

(compare with (3.6)) which solves the linear elliptic PDE (in the sense of distributions)

$$-\Delta H_N = \mathsf{c}_\mathsf{d}\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big) \tag{4.6}$$

and use for the first time crucially the Coulomb nature of the interaction to write

$$\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) \simeq -\frac{1}{\mathsf{c}_\mathsf{d}}\int_{\mathbb{R}^\mathsf{d}} H_N\Delta H_N = \frac{1}{\mathsf{c}_\mathsf{d}}\int_{\mathbb{R}^\mathsf{d}} |\nabla H_N|^2 \tag{4.7}$$

after integrating by parts by Green's formula. This computation is in fact incorrect because it ignores the diagonal terms which must be removed from the integral, and yields a divergent integral $\int|\nabla H_N|^2$ (it diverges near each point x_i of the configuration). However, this computation can be done properly by removing the infinite diagonal terms and "renormalizing" the infinite integral, replacing $\int|\nabla H_N|^2$ by

$$\int_{\mathbb{R}^\mathsf{d}} |\nabla H_{N,\eta}|^2 - N\mathsf{c}_\mathsf{d} g(\eta)$$

where we replace H_N by $H_{N,\eta}$, its "truncation" at level η (here $\eta = \alpha N^{-1/\mathsf{d}}$ with α a small fixed number) — more precisely $H_{N,\eta}$ is obtained by replacing the Dirac masses in (4.5) by uniform measures of total mass 1 supported on the sphere $\partial B(x_i,\eta)$ — and then removing the appropriate divergent part $\mathsf{c}_\mathsf{d}\mathsf{g}(\eta)$. The name *renormalized energy* originates in the work of Bethuel-Brezis-Hélein [23] in the context of two-dimensional Ginzburg-Landau vortices, where a similar (although different) renormalization procedure was introduced. Such a computation allows to replace the double integral, or sum of pairwise interactions of all the charges and "background", by a single integral, which is local in the potential H_N. This transformation is very useful, and uses crucially the fact that g is the kernel of a local operator (the Laplacian).

This electric energy $\int_{\mathbb{R}^\mathsf{d}}|\nabla H_{N,\eta}|^2$ is coercive and can thus serve to control the "fluctuations" $\sum_{i=1}^N \delta_{x_i} - N\mu_V$, in fact it is formally $\frac{1}{\mathsf{c}_\mathsf{d}}\|\nabla\Delta^{-1}(\sum_{i=1}^N \delta_{x_i} - N\mu_V)\|_{L^2}^2$. The relations (4.3)–(4.7)

can be inserted into the Gibbs measure (1.7) to yield so-called "concentration results" in the case with temperature, see [167] (for prior such concentration results, see [33, 46, 131]).

4.2. **Blow-up and limiting energy.** As we have seen, the configurations we are interested in are concentrated on (or near) the support of μ_V which is a set of macroscopic size and dimension d, and the typical distance between neighboring points is $N^{-1/\mathsf{d}}$. The next step is then to blow-up the configurations by $N^{1/\mathsf{d}}$ and take the $N \to \infty$ limit in $F_N^{\mu_V}$. This leads us to a renormalized energy that we define just below. It allows to compute a total Coulomb interaction for an infinite system of discrete point charges in a constant neutralizing background of fixed density 1. Such a system is often called a *jellium* in physics, and is sometimes considered as a toy model for matter, with a uniform electron sea and ions whose positions remain to be optimized.

From now on, we assume that Σ, the support of μ_V is a set with a regular boundary and $\mu_V(x)$ is a regular density function in Σ. Centering at some point x in Σ, we may blow-up the configuration by setting $x_i' = (N\mu_V(x))^{1/\mathsf{d}}\,(x_i - x)$ for each i. This way we expect to have a density of points equal to 1 after rescaling. Rescaling and taking $N \to \infty$ in (4.6), we are led to $H_N \to H$ with H solving an equation of the form

$$-\Delta H = \mathsf{c}_\mathsf{d}(\mathcal{C} - 1) \tag{4.8}$$

where $\mathcal{C}$ is a locally finite sum of Dirac masses.

Definition 4.1 ([144, 152, 160, 161])**.** The (Coulomb) renormalized energy of H is

$$\mathcal{W}(H) := \lim_{\eta \to 0} \mathcal{W}_\eta(H) \tag{4.9}$$

where we let

$$\mathcal{W}_\eta(H) := \limsup_{R\to\infty} \frac{1}{R^\mathsf{d}} \int_{[-\frac{R}{2},\frac{R}{2}]^\mathsf{d}} |\nabla H_\eta|^2 - \mathsf{c}_\mathsf{d}\mathsf{g}(\eta) \tag{4.10}$$

and H_η is a truncation of H performed similarly as above.
We define the renormalized energy of a point configuration $\mathcal{C}$ as

$$\mathbb{W}(\mathcal{C}) := \inf\{\mathcal{W}(H) \mid -\Delta H = \mathsf{c}_\mathsf{d}(\mathcal{C} - 1)\} \tag{4.11}$$

with the convention $\inf(\emptyset) = +\infty$.

It is not a priori clear how to define a total Coulomb interaction of such a jellium system, because of the infinite size of the system and because of its lack of local charge neutrality. The definitions we presented avoid having to go through computing the sum of pairwise interactions between particles (it would not even be clear how to sum them), but instead replace it with (renormalized variants of) the extensive quantity $\int |\nabla H|^2$.

The energy $\mathbb{W}$ can be proven to be bounded below and to have a minimizer; moreover, its minimum can be achieved as the limit of energies of periodic configurations (with larger and larger period), for all these aspects see for instance [169].

4.3. **Cristallization questions for minimizers.** Determining the value of $\min \mathbb{W}$ is an open question, with the exception of the one-dimensional analogues for which the minimum is achieved at the lattice $\mathbb{Z}$ [120, 160].

The only question that we can answer so far is that of the minimization over the restricted class of pure lattice configurations in dimension $\mathsf{d} = 2$, i.e. configurations which are exactly a lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ with $det(\vec{u}, \vec{v}) = 1$.

Theorem 3. *The minimum of $\mathbb{W}$ over lattices of volume 1 in dimension 2 is achieved uniquely by the triangular lattice.*

Here the triangular lattice means $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$, properly scaled, i.e. what is called the Abrikosov lattice in the context of superconductivity. This result is essentially equivalent (see [48, 139]) to a result on the minimization of the Epstein ζ function of the lattice

$$\zeta_s(\Lambda) := \sum_{p \in \Lambda \setminus \{0\}} \frac{1}{|p|^s}$$

proven in the 50's by Cassels, Rankin, Ennola, Diananda, cf. [137] and references therein. It corresponds to the minimization of the "height" of flat tori, in the sense of Arakelov geometry.

One may ask whether this triangular lattice does achieve the global minimum of $\mathbb{W}$. The fact that the Abrikosov lattice is observed in superconductors, combined with the fact that $\mathbb{W}$ can be derived as the limiting minimization problem of Ginzburg-Landau [159], justify conjecturing this.

Conjecture 4.2. The triangular lattice is a global minimizer of $\mathbb{W}$ in dimension 2.

It was also recently proven in [20] that this conjecture is equivalent to a conjecture of Brauchart-Hardin-Saff [37] on the next order term in the asymptotic expansion of the minimal logarithmic energy on the sphere (an important problem in approximation theory, also related to Smale's "7th problem for the 21st century"), which is obtained by formal analytic continuation, hence by very different arguments.

Note that the triangular lattice is also conjectured to have universally minimizing properties [52] i.e. to be the minimizer for a broad class of interactions. An analogous role is played in dimensions 8 and 24 by the E_8 and Leech lattices, respectively, which provide the solutions to the best packing problem in [53, 187].

In dimension $\mathsf{d} \geq 3$ the minimization of $\mathbb{W}$ even restricted to the class of lattices is an open question, except in dimensions 4, 8 and 24 where a strict local minimizer is known [163] (E_8 and Leech in dimensions 8 and 24). Similarly, one may conjecture that in low dimensions, the minimum of $\mathbb{W}$ is achieved by some particular lattice. In large dimensions, lattices are not expected to be minimizing.

These questions belongs to the more general family of crystallization problems, see [27] for a review. A typical such question is, given an interaction kernel g in any dimension, to determine the point positions that minimize

$$\sum_{i \neq j} \mathsf{g}(x_i - x_j)$$

(with some kind of boundary condition), or rather

$$\lim_{R \to \infty} \frac{1}{|B_R|} \sum_{i \neq j, x_i, x_j \in B_R} \mathsf{g}(x_i - x_j),$$

and to determine whether the minimizing configurations are perfect lattices. Such questions are fundamental in order to understand the cristalline structure of matter. One should immediately stress that there are very few positive results in that direction in the literature (in fact it is very rare to have a proof that the solution to some minimization problem is periodic, except in dimension 1). Some exceptions include the two-dimensional sphere packing problem and an extension of Radin's proof [146] by Theil [183] for a class of very short range Lennard-Jones potentials.

4.4. **Convergence results for minimizers.** Given a (sequence of) configuration(s) $(x_1,\dots,x_N)$, we examine as mentioned before the blow-up point configurations $\{(\mu_V(x)N)^{1/d}(x_i-x)\}$ and their infinite limits $\mathcal{C}$. We also need to let the blow-up center x vary over Σ, the support of μ_V. Averaging near the blow-up center x yields a "point process" P_N^x: a point process is precisely defined as a probability distribution on the space of possibly infinite point configurations, denoted Config. Here the point process P_N^x is essentially the Dirac mass at the blown-up configuration $\{(\mu_V(x)N)^{1/d}(x_i-x)\}$. This way, we form a "tagged point process" P_N (where the tag is the memory of the blow-up center), probability on $\Sigma\times$ Config, whose "slices" are the P_N^x. Taking limits $N\to\infty$ (up to subsequences), we obtain limiting tagged point processes P, which are all stationary, i.e. translation-invariant. We may also define the renormalized Coulomb energy at the level of tagged point processes as

$$\overline{\mathbb{W}}(P):=\frac{1}{2\mathsf{c}_\mathsf{d}}\int_\Sigma\int \mathbb{W}(\mathcal{C})dP^x(\mathcal{C})dx.$$

In view of (4.3) and the previous discussion, we may expect the following informally stated result (which we state only in the Coulomb cases, for extensions to (1.5) see [160] and to (1.6) see [144]).

Theorem 4 ([152, 161]). *Consider configurations such that*

$$\mathcal{H}_N(x_1,\dots,x_N)-N^2\mathcal{I}_V(\mu_V)\le CN^{2-\frac{2}{\mathsf{d}}}.$$

Then up to extraction P_N converges to some P and

$$\mathcal{H}_N(x_1,\dots,x_N)\simeq N^2\mathcal{I}_V(\mu_V)+N^{2-\frac{2}{\mathsf{d}}}\overline{\mathbb{W}}(P)+o(N^{2-\frac{2}{\mathsf{d}}}) \tag{4.12}$$

[4] *and in particular*

$$\min\mathcal{H}_N=N^2\mathcal{I}_V(\mu_V)+N^{2-\frac{2}{\mathsf{d}}}\min\overline{\mathbb{W}}+o(N^{2-\frac{2}{\mathsf{d}}}). \tag{4.13}$$

Since $\overline{\mathbb{W}}$ is an average of $\mathbb{W}$, the result (4.13) can be read as: after suitable blow-up around a point x, for a.e. $x\in\Sigma$, the minimizing configurations converge to minimizers of $\mathbb{W}$. If one believes minimizers of $\mathbb{W}$ to ressemble lattices, then it means that minimizers of $\mathcal{H}_N$ should do so as well. In any case, $\mathbb{W}$ can distinguish between different lattices (in dimension 2, the triangular lattice has less energy than the square lattice) and we expect $\mathbb{W}$ to be a good quantitative measure of disorder of a configuration (see [32]).

The analogous result was proven in [159] for the vortices in minimizers of the Ginzburg-Landau energy (2.1): they also converge after blow-up to minimizers of $\mathbb{W}$, providing a first rigorous justification of the Abrikosov lattice observed in experiments, modulo Conjecture 4.2. The same result was also obtained in [87] for a two-dimensional model of small charged droplets interacting logarithmically called the Ohta-Kawasaki model – a sort of variant of Gamov's liquid drop model, after the corresponding mean-field limit results was established in [86].

One advantage of the above theorem is that it is valid for generic configurations and not just for minimizers. When using the minimality, better "rigidity results" (as alluded to above) of minimizers can be proven: points are separated by $\frac{C}{(N\|\mu_V\|_\infty)^{1/d}}$ for some fixed $C>0$ and there is uniform distribution of points and energy, down to the microscopic scale, see [138, 143, 144].

Theorem 4 relies on two ingredients which serve to prove respectively a lower bound and an upper bound for the next-order energy. The first is a general method for proving lower

[4] In dimension $\mathsf{d}=2$, there is an additional additive term $\frac{N}{4}\log N$ in both relations

bounds for energies which have two instrinsic scales (here the macroscopic scale 1 and the microscopic scale $N^{-1/\mathsf{d}}$) and which is handled via the introduction of the probability measures on point patterns P_N described above. This method (see [161, 169]), inspired by Varadhan, is reminiscent of Young measures and of [4]. The second is a "screening procedure" which allows to exploit the local nature of the next-order energy expressed in terms of H_N, to paste together configurations given over large microscopic cubes and compute their next-order energy additively. To do so, we need to modify the configuration in a neighborhood of the boundary of the cube so as to make the cube neutral in charge and to make ∇H_N tangent to the boundary. This effectively screens the configuration in each cube in the sense that it makes the interaction between the different cubes vanish, so that the energy $\int |\nabla H_N|^2$ becomes proportional to the volume. One needs to show that this modification can be made while altering only a negligible fraction of the points and a negligible amount of the energy. This construction is reminiscent of [3]. It is here crucial that the interaction is Coulomb so that the energy is expressed by a local function of H_N, which itself solves an elliptic PDE, making it possible to use the toolbox on estimates for such PDEs.

The next order study has not at all been touched in the case of dynamics, but it has been tackled in the statistical mechanics setting of (1.7).

4.5. **Next-order with temperature.** Here the interesting temperature regime (to see non-trivial temperature effects) turns out to be $\beta_N = \beta N^{\frac{2}{\mathsf{d}}-1}$.

In contrast to the macroscopic result, several observations (e.g. by numerical simulation, see Figure 3) suggest that the behavior of the system at the microscopic scale depends heavily on β, and one would like to describe this more precisely. In the particular case of (1.5) or

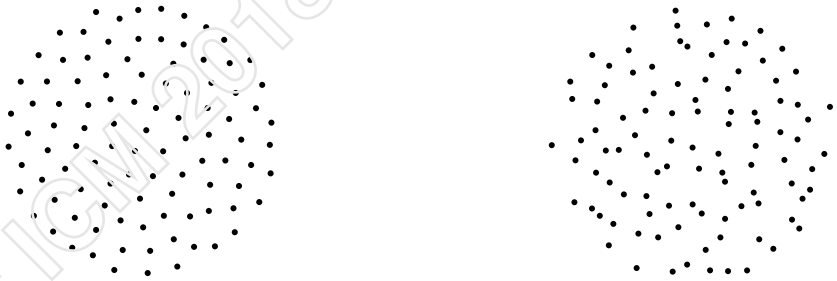

FIGURE 3. Case Log2 with $N = 100$ and $V(x) = |x|^2$, for $\beta = 400$ (left) and $\beta = 5$ (right).

(1.2) with $\beta = 2$, which both arise in Random Matrix Theory, many things can be computed explicitly, and expansions of $\log Z_{N,\beta}$ as $N \to \infty$, Central Limit Theorems for linear statistics, universality in V (after suitable rescaling) of the microscopic behavior and local statistics of the points, are known [12, 14, 33–36, 107, 173]. Generalizing such results to higher dimensions and all β's is a significant challenge.

4.6. **Large Deviations Principle.** A first approach consists in following the path taken for minimizers and using the next-order expansion of $\mathcal{H}_N$ given in (4.12). This expansion can

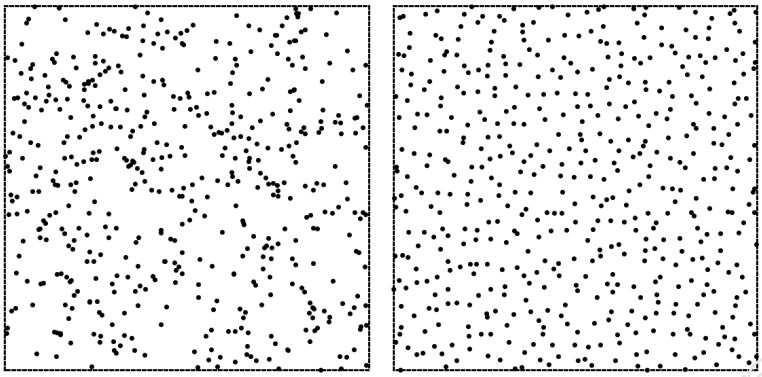

FIGURE 4. Simulation of the Poisson point process with intensity 1 (left), and the Ginibre point process with intensity 1 (right)

be formally inserted into (1.7), however this is not sufficient: to get a complete result, one needs to understand precisely how much volume in configuration space $(\mathbb{R}^{\mathsf{d}})^N$ is occupied near a given tagged point process P — this will give rise to an entropy term — and how much error (in both volume and energy) the screening construction creates. At the end we obtain a Large Deviations Principle expressed at the level of the microscopic point processes P, instead of the macroscopic empirical measures μ in Theorem 2. This is sometimes called "type-III large deviations" or large deviations at the level of empirical fields. Such results can be found in [186], [69], the relative specific entropy that we will use is formalized in [70] (for the non-interacting discrete case), [80] (for the interacting discrete case) and [81] (for the interacting continuous case).

To state the result precisely, we need to introduce the Poisson point process with intensity 1, denoted Π, as the point process characterized by the fact that for any bounded Borel set B in $\mathbb{R}^{\mathsf{d}}$

$$\Pi\left(N(B)=n\right)=\frac{|B|^n}{n!}e^{-|B|}$$

where $N(B)$ denotes the number of points in B. The expectation of the number of points in B can then be computed to be $|B|$, and one also observes that the number of points in two disjoint sets are independent, thus the points "don't interact", see Figure 4 for a picture. The "specific" relative entropy $\overline{\mathsf{ent}}$ with respect to Π refers to the fact that it has to be computed taking an infinite volume limit, see [147] for a precise definition. One can just think that it measures how close the point process is to the Poisson one.

For any $\beta>0$, we then define a free energy functional $\overline{\mathcal{F}}_\beta$ as

$$\overline{\mathcal{F}}_\beta(P):=\frac{\beta}{2}\overline{\mathbb{W}}(P)+\overline{\mathsf{ent}}[P|\Pi]. \tag{4.14}$$

Theorem 5 ([121]). *Under suitable assumptions, for any $\beta>0$ a Large Deviations Principle at speed N with good rate function $\overline{\mathcal{F}}_\beta-\inf\overline{\mathcal{F}}_\beta$ holds in the sense that*

$$\mathbb{P}_{N,\beta}(P_N\simeq P)\simeq e^{-N(\overline{\mathcal{F}}_\beta(P)-\inf\overline{\mathcal{F}}_\beta)}$$

This way, the the Gibbs measure $\mathbb{P}_{N,\beta}$ concentrates on microscopic point processes which minimize $\overline{\mathcal{F}}_\beta$. This minimization problem corresponds to some balancing (depending on

β) between $\overline{\mathbb{W}}$, which prefers order of the configurations (and expectedly crystallization in low dimensions), and the relative entropy term which measures the distance to the Poisson process, thus prefers microscopic disorder and decorrelation between the points. As $\beta \to 0$, or temperature gets very large, the entropy term dominates and one can prove [120] that the minimizer of $\overline{\mathcal{F}}_\beta$ converges to the Poisson process. On the contrary, when $\beta \to \infty$, the $\mathbb{W}$ term dominates, and prefers regular and rigid configurations. (In the case (1.5) where the minimum of $\mathbb{W}$ is known to be achieved by the lattice, this can be made into a complete proof of crystallization as $\beta \to \infty$, cf. [120]). When β is intermediate then both terms are important and one does not expect crystallization in that sense nor complete decorrelation. For separation results analogous to those quoted about minimizers, one may see [7] and references therein.

The existence of a minimizer to $\overline{\mathcal{F}}_\beta$ is known, it is certainly nonunique due to the rotational invariance of the problem, but it is not known whether it is unique modulo rotations, nor is the existence of a limiting point process P (independent of the subsequence) in general. The latter is however known to exist in certain ensembles arising in random matrix theory: for (1.5) for any β, it is the so-called sine-β process [114, 185], and for (1.2) for $\beta = 2$ and V quadratic, it is the Ginibre point process [83], shown in Figure 4. It was also shown to exist for the jellium for small β in [99]. A consequence of Theorem 5 is to provide a variational interpretation to these point processes. One may hope to understand phase-transitions at the level of these processes, possibly via this variational interpretation, however this is completely open. While in dimension 1, the point process is expected to always be unique, in dimension 2, phase-transitions and symmetry breaking in positional or orientational order may happen. One would also like to understand the decay of the two-point correlation function and its possible change in rate, corresponding to a phase-transition. In the one-dimensional logarithmic case, the limits of the correlation functions are computed for rational β's [71] and indicate a phase-transition.

A second corollary obtained as a by-product of Theorem 5 is the existence of a next order expansion of the free energy $-\beta^{-1}\log Z_{N,\beta}$.

Corollary 4.3 ([121]).

$$-\beta^{-1}\log Z_{N,\beta} = N^{1+\frac{2}{\mathsf{d}}}\mathcal{I}_V(\mu_V) + N\min\overline{\mathcal{F}}_\beta + o(N) \tag{4.15}$$

in the cases (1.3)*; and in the cases* (1.2), (1.5),

$$-\beta^{-1}\log Z_{N,\beta} = N^2\mathcal{I}_V(\mu_V) - \frac{N}{2\mathsf{d}}\log N + N\min\overline{\mathcal{F}}_\beta + o(N)$$

or more explicitly

$$-\beta^{-1}\log Z_{N,\beta} = N^2\mathcal{I}_V(\mu_V) - \frac{N}{2\mathsf{d}}\log N + NC_\beta + N\left(\frac{1}{\beta} - \frac{1}{2\mathsf{d}}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx + o(N), \tag{4.16}$$

where C_β depends only on β, but not on V.

This formulae are to be compared with the results of [12, 33, 34, 173] in the Log1 case, the semi-rigorous formulae in [190] in the dimension 2 Coulomb case, and are the best-known information on the free energy otherwise. We recall that understanding the free energy is fundamental for the description of the properties of the system. For instance, the explicit dependence in V exhibited in (4.16) will be the key to proving the result of the next section.

Finally, note that a similar result to the above theorem and corollary can be obtained in the case of the two-dimensional two-component plasma alluded to in Section 2.4, see [122].

4.7. **A Central Limit Theorem for fluctuations.** Another approach to understanding the rigidity of configurations and how it depends on the temperature is to examine the behavior of the linear statistics of the fluctuations, i.e. consider, for a regular test function f, the quantity

$$\sum_{i=1}^N f(x_i) - N\int f d\mu_V.$$

Theorem 6 ([168]). *In the case (1.2), assume $V \in C^4$ and the previous assumptions on μ_V and $\partial\Sigma$, and let $f \in C^4_c(\mathbb{R}^2)$ or $C^3_c(\Sigma)$. If Σ has $m \geq 2$ connected components Σ_i, add $m-1$ conditions $\int_{\partial\Sigma_i} \Delta f^\Sigma = 0$ where f^Σ is the harmonic extension of f outside Σ. Then*

$$\sum_{i=1}^N f(x_i) - N\int_\Sigma f\, d\mu_V$$

converges in law as $N \to \infty$ to a Gaussian distribution with

$$\text{mean} = \frac{1}{2\pi}\left(\frac{1}{\beta}-\frac{1}{4}\right)\int_{\mathbb{R}^2} \Delta f\,(\mathbf{1}_\Sigma + \log \Delta V)^\Sigma \qquad \text{variance} = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2} |\nabla f^\Sigma|^2.$$

The result can moreover be localized with f supported on any mesoscale $N^{-\alpha}$, $\alpha < \frac{1}{2}$, and it is true as well for energy minimizers, taking formally $\beta = \infty$.

This result can be interpreted in terms of the convergence of H_N (of (4.5)) to a suitable so-called "Gaussian Free Field", a sort of two-dimensional analogue of Brownian motion. This theorem shows that if f is smooth enough, the fluctuations of linear statistics are typically of order 1, i.e. much smaller than the sum of N iid random variables which is typically or order $\sqrt{N}$. This a manifestation of rigidity, which even holds down to the mesoscales. Note that the regularity of f is necessary, the result is false if f is discontinuous, however the precise threshhold of regularity is not known.

In dimension 1, this theorem was first proven in [107] for polynomial V and f analytic. It was later generalized in [13, 14, 33, 34, 173, 188]. In dimension 2, this result was proven for the determinantal case $\beta = 2$ in [148] (for V quadratic) and [8] under analyticity assumptions. It was then proven for all β simultaneously as [168] in [11], with f assumed to be supported in Σ.

The approach for proving such results has generally been based on Dyson-Schwinger (or "loop") equations. If the extra conditions do not hold, then the CLT is not expected to hold. Rather, the limit should be a Gaussian convolved with a discrete Gaussian variable, as shown in the `Log1` case in [34].

To prove Theorem 6, following the approach pioneered by Johansson [107], we compute the Laplace transform of these linear statistics and see that it reduces to understanding the ratio of two partition functions, the original one and that of a Coulomb gas with potential V replaced by $V_t = V + tf$ with t small. Thanks to [171] the variation of the equilibrium measure associated to this replacement is well understood. We are then able to leverage on the expansion of the partition function of (4.16) to compute the desired ratio, using also a change of variables which is a transport map between the equilibrium measure μ_V and the

perturbed equilibrium measure. Note that the use of changes of variables in this context is not new, cf. [12, 33, 107, 173]. In our approach, it essentially replaces the use of the loop or Dyson-Schwinger equations.

4.8. **More general interactions.** It remains to understand how much of the behavior we described are really specific to Coulomb interactions. Already Theorems 4 and 5 were shown in [121, 144] to hold for the more general Riesz interactions with $\mathsf{d} - 2 \leq s < \mathsf{d}$. This is thanks to the fact that the Riesz kernel is the kernel for a fractional Laplacian, which is not a local operator but can be interpreted as a local operator after adding one spatial dimension, according to the procedure of Caffarelli-Silvestre [40]. The results of Theorem 5 are also valid in the *hypersingular* Riesz interactions $\mathsf{s} > \mathsf{d}$ (see [92]), where the kernel is very singular but also decays very fast. The Gaussian behavior of the fluctuations seen in Theorem 6 is for now proved only in the logarithmic cases, but it remains to show whether it holds for more general Coulomb cases and even possibly more general interactions as well.

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